MONODROMY OF SOLUTIONS OF THE ELLIPTIC QUANTUM KNIZHNIK-ZAMOLODCHIKOV-BERNARD DIFFERENCE EQUATIONS

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Abstract. The elliptic quantum Knizhnik–Zamolodchikov–Bernard (qKZB) difference equations associated to the elliptic quantum group $E_{\tau,\eta}(sl_2)$ is a system of difference equations with values in a tensor product of representations of the quantum group and defined in terms of the elliptic R-matrices associated with pairs of representations of the quantum group. In this paper we solve the qKZB equations in terms of elliptic hypergeometric functions and decribe the monodromy properties of solutions. It turns out that the monodromy transformations of solutions are described in terms of elliptic R-matrices associated with pairs of representations of the "dual" elliptic quantum group $E_{p,\eta}(sl_2)$, where p is the step of the difference equations. Our description of the monodromy is analogous to the Kohno-Drinfeld description the monodromy group of solutions of the KZ differential equations associated to a simple Lie algebra in terms of the corresponding quantum group.

1. Introduction

In this paper we solve the system of elliptic quantum Knizhnik–Zamolodchikov–Bernard (qKZB) difference equations associated with the elliptic quantum group $E_{\tau,\eta}(sl_2)$ and describe the monodromy properties of solutions.

The qKZB equations [F] are a quantum deformation of the KZB differential equations obeyed by correlation functions of the Wess–Zumino–Witten model on tori. The qKZB equations have the form

$$\Psi(z_1,\dots,z_j+p,\dots,z_n)=K_j(z_1,\dots,z_n;\tau,\eta,p)\Psi(z_1,\dots,z_n).$$

The unknown function Ψ takes values in a space of vector valued functions of a complex variable λ, and the K_j are difference operators in λ. The parameters of this system of equations are τ (the period of the elliptic curve), η ("Planck's constant"), p (the step) and n "highest weights" $\Lambda_1,\dots,\Lambda_n\in\mathbb{C}$. The operators K_j are expressed in terms of R-matrices of the elliptic quantum group $E_{\tau,\eta}(sl_2)$.

In the trigonometric limit $\tau\to i\infty$, the qKZB equations reduce to the trigonometric qKZ equations [FR] obeyed by correlation functions of statistical models and form factors of integrable quantum field theories in 1+1 dimensions [JM, S].

The KZB equations can be obtained in the semiclassical limit: $\eta\to 0$, $p\to 0$, p/η finite.

When the step p of the qKZB equations goes to zero (with the other parameters fixed) our construction gives common eigenfunctions of the n commuting operators $K_j(z_1,\dots,z_n;\tau,\eta,0)$ in the form of the Bethe

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ansatz [FTV]. These difference operators are closely related to the transfer matrices of IRF models of statistical mechanics [F, FV2].

Our first main result is a construction of solutions of the qKZB equations in the form of multidimensional elliptic hypergeometric intergals.

Our second main result is a description of the monodromy properties of solutions. We show that the qKZB equations can be considered as difference equations on the product of several copies of the elliptic curve with modulus τ with values in a suitable vector bundle. Therefore, a natural question is to describe the monodromy of solutions with respect to shifts of arguments of solutions by periods of the elliptic curve. It turns out that the monodromy transformations of solutions are described in terms of R-matrices associated with pairs of representations of the "dual" elliptic quantum group $E_{p,\eta}(sl_2)$, where p is the step of the difference equations.

Our description of the monodromy is analogous to the Kohno-Drinfeld description [K, D] of the monodromy group of solutions of the KZ differential equations associated to a simple Lie algebra in terms of the corresponding quantum group.

The results of this paper are parallel to the results on solutions of the rational and trigonometric qKZ equations of [V, TV1, TV2], which are based on the representation theory of the Yangian $Y(sl_2)$ and the affine quantum universal enveloping algebra $U_q(\widehat{sl_2})$, respectively. In particular, in [TV1] the monodromy of the qKZ difference equations associated with the Yangian $Y(sl_2)$ is descibed in terms of the affine quantum group $U_q(\widehat{sl_2})$, where the parameter q is connected with the step p of the equations by $q = e^{\pi i/p}$. In [TV2] the monodromy of the qKZ difference equations associated with the affine quantum group $U_q(\widehat{sl_2})$ is descibed in terms of the elliptic quantum group $E_{\tau,\eta}(sl_2)$ where the parameters τ and η of the elliptic quantum group are connected with the parameters q and the multiplicative step p of the equations by relations $p = e^{2\pi i\tau}$ and $q = e^{-2\pi i\eta}$.

The paper is organized as follows. We begin by introducing the notion of R-matrices and the qKZB equations in Section 2. The geometric construction of R-matrices is given in Section 3. At the end of that section we show how to obtain representations of $E_{\tau,\eta}(sl_2)$ in this way, and give some explicit formulae for R-matrix elements.

In Section 4 we describe transformation properties of the qKZB equations with respect to shifts of the arguments by τ and 1 and show that the qKZB equations can be considered as difference equations on a product of several copies of the elliptic curve with modulus τ.

In Section 5 we describe formal integral representations of solutions of the qKZB equations. We constuct solutions of the qKZB equations in Section 6. The monodromy properties of solutions are described in Section 7.

2. R-matrices, qKZB equations

2.1. **R-matrices.** The qKZB equations are given in terms of R-matrices of elliptic quantum groups. In the sl_2 case, these R-matrices have the following properties. Let $\mathfrak{h} = \mathbb{C}h$ be a one-dimensional Lie algebra with generator h. For each $\Lambda \in \mathbb{C}$ consider the $\mathfrak{h}$-module $V_\Lambda = \oplus_{j=0}^{\infty}\mathbb{C}e_j$, with $he_j = (\Lambda - 2j)e_j$. For each pair Λ_1, Λ_2 of complex numbers we have a meromorphic function, called the R-matrix, $R_{\Lambda_1,\Lambda_2}(z,\lambda)$ of two complex variables, with values in $\mathrm{End}\,(V_{\Lambda_1} \otimes V_{\Lambda_2})$.

The main properties of the R-matrices are

I. The zero weight property: for any Λ_i, z, λ, $[R_{\Lambda_1,\Lambda_2}(z,\lambda), h^{(1)} + h^{(2)}] = 0$.

II. For any $\Lambda_1, \Lambda_2, \Lambda_3$, the dynamical Yang–Baxter equation

$$\begin{aligned} &R_{\Lambda_1,\Lambda_2}(z,\lambda - 2\eta h^{(3)})^{(12)} R_{\Lambda_1,\Lambda_3}(z+w,\lambda)^{(13)} R_{\Lambda_2,\Lambda_3}(w,\lambda - 2\eta h^{(1)})^{(23)} \\ &\quad = R_{\Lambda_2,\Lambda_3}(w,\lambda)^{(23)} R_{\Lambda_1,\Lambda_3}(z+w,\lambda - 2\eta h^{(2)})^{(13)} R_{\Lambda_1,\Lambda_2}(z,\lambda)^{(12)}, \end{aligned}$$

holds in $\mathrm{End}\,(V_{\Lambda_1} \otimes V_{\Lambda_2} \otimes V_{\Lambda_3})$ for all z, w, λ.

III. For all Λ_1, Λ_2, z, λ, $R_{\Lambda_1,\Lambda_2}(z,\lambda)^{(12)} R_{\Lambda_2,\Lambda_1}(-z,\lambda)^{(21)} = \mathrm{Id}$. This property is called the "unitarity".

We use the following notation: if $X \in \mathrm{End}\,(V_i)$, then we denote by $X^{(i)} \in \mathrm{End}\,(V_1 \otimes \cdots \otimes V_n)$ the operator $\cdots \otimes \mathrm{Id} \otimes X \otimes \mathrm{Id} \otimes \cdots$, acting non-trivially on the ith factor of a tensor product of vector spaces, and if $X = \sum X_k \otimes Y_k \in \mathrm{End}\,(V_i \otimes V_j)$, then we set $X^{(ij)} = \sum X_k^{(i)} Y_k^{(j)}$. If $X(\mu_1, \ldots, \mu_n)$ is a function with values in $\mathrm{End}\,(V_1 \otimes \cdots \otimes V_n)$, then $X(h^{(1)}, \ldots, h^{(n)})v = X(\mu_1, \ldots, \mu_n)v$ if $h^{(i)}v = \mu_i v$, for all $i = 1, \ldots, n$.

For each τ in the upper half plane and generic $\eta \in \mathbb{C}$ ("Planck's constant") a system of R-matrices $R_{\Lambda_1,\Lambda_2}(z,\lambda)$ obeying I – III was constructed in [FV1]. They are characterized by an intertwining property

with respect to the action of the elliptic quantum group $E_{\tau,\eta}(sl_2)$ on tensor products of evaluation Verma modules.

2.2. **qKZB equations.** Fix the parameters τ, η. Fix also n complex numbers $\Lambda_1, \dots, \Lambda_n$ and an additional parameter $p \in \mathbb{C}$. Let $V = V_{\Lambda_1} \otimes \cdots \otimes V_{\Lambda_n}$. The kernel of $h^{(1)} + \cdots + h^{(n)}$ on V is called the zero-weight space and is denoted $V[0]$. More generally, we write $V[\mu]$ for the eigenspace of $\sum h^{(i)}$ with eigenvalue μ. The qKZB equations are difference equations for a function $\Psi(z_1, \dots, z_n, \lambda)$ of n complex variables $z_1, \dots, z_n$ with values in the space of meromorphic functions $\mathrm{Fun}\,(V[0])$ of a complex variable λ with values in $V[0]$.

The qKZB equations [F] have the form

$$\begin{aligned}\Psi(z_1, \dots, z_j + p, \dots, z_n) = R_{j,j-1}(z_j - z_{j-1} + p) \cdots R_{j,1}(z_j - z_1 + p) \\ \Gamma_j R_{j,n}(z_j - z_n) \cdots , R_{j,j+1}(z_j - z_{j+1}) \Psi(z_1, \dots, z_n)\end{aligned} \qquad (1)$$

Here $R_{k,l}(z)$ is the operator of multiplication by

$$R_{\Lambda_k,\Lambda_l}(z, \lambda - 2\eta \sum_{\substack{j=1 \\ j\neq k}}^{l-1} h^{(j)})^{(k,l)}$$

acting on the kth and lth factor of the tensor product, and Γ_j is the linear difference operator such that $\Gamma_j \Psi(\lambda) = \Psi(\lambda - 2\eta\mu)$ if $h^{(j)}\Psi = \mu\Psi$.

The consistency of these equations follows from I–III. In other words, the qKZB equations may be viewed as the equation of horizontality for a flat discrete connection on a trivial vector bundle with fiber $\mathrm{Fun}\,(V[0])$ over an open subset of $\mathbb{C}^n$.

2.3. **Finite-dimensional representations.** If Λ is a nonnegative integer, V_Λ contains the subspace $SV_\Lambda = \oplus_{j=\Lambda+1}^{\infty} \mathbb{C} e_j$ with the property that, for any M, $SV_\Lambda \otimes V_{\mathrm{M}}$ and $V_{\mathrm{M}} \otimes SV_\Lambda$ are preserved by the R-matrices $R_{\Lambda,\mathrm{M}}(z, \lambda)$ and $R_{\mathrm{M},\Lambda}(z, \lambda)$, respectively, see [FV1] and Theorem 6. Let $L_\Lambda = V_\Lambda / SV_\Lambda$, $\Lambda \in \mathbb{Z}_{\geqslant 0}$. Then, in particular, for any nonnegative integers Λ and M, $R_{\Lambda,\mathrm{M}}(z, \lambda)$ induces a map, also denoted by $R_{\Lambda,\mathrm{M}}(z, \lambda)$, on the finite-dimensional space $L_\Lambda \otimes L_{\mathrm{M}}$.

The simplest nontrivial case is $\Lambda = \mathrm{M} = 1$. Then $R_{1,1}(z, \lambda)$ is defined on a four-dimensional vector space and coincides with the *fundamental* R-matrix, the matrix of structure constants of the elliptic quantum group $E_{\tau,\eta}(sl_2)$.

In any case, if $\Lambda_1, \dots, \Lambda_n$ are nonnegative integers, we can consider the qKZB equations (1) on functions with values in the zero weight space of $L_{\Lambda_1} \otimes \cdots \otimes L_{\Lambda_n}$.

The results of this paper obtained for the solutions with values in $\otimes_j V_{\Lambda_j}$ extend to this case: let $\pi : \otimes_{j=1}^n V_{\Lambda_j} \to \otimes_{j=1}^n L_{\Lambda_j}$ denote the canonical projection.

Lemma 1. *Let $\Psi(z_1, \dots, z_n)$ be a solution of the qKZB equations with values in $V[0] = V_{\Lambda_1} \otimes \cdots \otimes V_{\Lambda_n}[0]$. Then $\pi \circ \Psi(z_1, \dots, z_n)$ is a solution of the qKZB equations with values in $L[0] = L_{\Lambda_1} \otimes \cdots \otimes L_{\Lambda_n}[0]$.*

3. Modules over the elliptic quantum group as function spaces

In this section we realize the spaces dual to tensor products of evaluation Verma modules over $E_{\tau,\eta}(sl_2)$ as spaces of functions. The R-matrices are then constructed geometrically.

Let us fix complex parameters τ, η with $\mathrm{Im}\,\tau > 0$, and complex numbers $\Lambda_1, \dots, \Lambda_n$. We set $a_i = \eta\Lambda_i$, $i = 1, \dots, n$.

3.1. **A space of symmetric functions.** Introduce a space of functions with an action of the symmetric group. Recall that the Jacobi theta function

$$\theta(t) = -\sum_{j\in\mathbb{Z}} e^{\pi i (j+\frac{1}{2})^2 \tau + 2\pi i (j+\frac{1}{2})(t+\frac{1}{2})},$$

has multipliers -1 and $-\exp(-2\pi i t - \pi i \tau)$ as $t \to t+1$ and $t \to t+\tau$, respectively. It is an odd entire function whose zeros are simple and lie on the lattice $\mathbb{Z} + \tau\mathbb{Z}$. It has the product formula

$$\theta(t) \;=\; 2e^{\pi i\tau/4} \sin(\pi t) \prod_{j=1}^{\infty} (1 - q^j)(1 - q^j e^{2\pi i t})(1 - q^j e^{-2\pi i t}), \qquad q = e^{2\pi i \tau}.$$

Definition: For complex numbers $a_1, \dots, a_n,\ z_1, \dots, z_n,\ \lambda$, let $\tilde{F}^m_{a_1,\dots,a_n}(z_1, \dots, z_n, \lambda)$ be the space of meromorphic functions $f(t_1, \dots, t_m)$ of m complex variables such that

(i) $\prod_{i<j} \theta(t_i - t_j + 2\eta) \prod_{i=1}^m \prod_{k=1}^n \theta(t_i - z_k - a_k) f$ is a holomorphic function on $\mathbb{C}^m$.

(ii) f is periodic with period 1 in each of its arguments and

$$f(\cdots, t_j + \tau, \cdots) = e^{-2\pi i(\lambda + 4\eta j - 2\eta)} f(\cdots, t_j, \cdots),$$

for all $j = 1, \ldots, m$.

The symmetric group S_m acts on $\tilde{F}^m_{a_1,\ldots,a_n}(z_1, \ldots, z_n, \lambda)$ so that the transposition of j and $j+1$ acts as

$$s_j f(t_1, \ldots, t_m) = f(t_1, \ldots, t_{j+1}, t_j, \ldots, t_m) \frac{\theta(t_j - t_{j+1} - 2\eta)}{\theta(t_j - t_{j+1} + 2\eta)}.$$

Definition: For any $m \in \mathbb{Z}_{>0}$, let $F^m_{a_1,\ldots,a_n}(z_1, \ldots, z_n, \lambda) = \tilde{F}^m_{a_1,\ldots,a_n}(z_1, \ldots, z_n, \lambda)^{S_m}$ be the space of S_m-invariant functions. If $m = 0$, then we set $F^0_{a_1,\ldots,a_n}(z_1, \ldots, z_n, \lambda) = \mathbb{C}$. We denote by Sym the symmetrization operator Sym $= \sum_{s \in S_m} s : \tilde{F}^m \to F^m$. Also, we set

$$F_{a_1,\ldots,a_n}(z_1, \ldots, z_n, \lambda) = \oplus_{m=0}^{\infty} F^m_{a_1,\ldots,a_n}(z_1, \ldots, z_n, \lambda),$$

and define an $\mathfrak{h}$-module structure on $F_{a_1,\ldots,a_n}(z_1, \ldots, z_n, \lambda))$ by letting h act by

$$h|_{F^m_{a_1,\ldots,a_n}(z_1,\ldots,z_n,\lambda)} = (\sum_{i=1}^{n} \Lambda_i - 2m)\mathrm{Id}, \qquad a_i = \eta \Lambda_i.$$

Clearly, $F^m_{a_{\sigma(1)},\ldots,a_{\sigma(n)}}(z_{\sigma(1)}, \ldots, z_{\sigma(n)}) = F^m_{a_1,\ldots,a_n}(z_1, \ldots, z_n, \lambda)$ for any permutation $\sigma \in S_n$.

It is shown in [FTV] that $F^m_{a_1,\ldots,a_n}(z_1, \ldots, z_n, \lambda)$ is a finite-dimensional vector space of dimension $\binom{n+m-1}{m}$.

Example: Let $n = 1$. Then $F^m_a(z, \lambda)$ is a one-dimensional space spanned by

$$\omega_m(t_1, \ldots, t_m, \lambda; z) = \prod_{i<j} \frac{\theta(t_i - t_j)}{\theta(t_i - t_j + 2\eta)} \prod_{j=1}^{m} \frac{\theta(\lambda + 2\eta m + t_j - z - a)}{\theta(t_j - z - a)}. \tag{2}$$

3.2. **Tensor products.**

Proposition 2. [FTV] *Let $n = n' + n''$, $m = m' + m''$ be nonnegative integers and $a_1, \ldots, a_n$, $z_1, \ldots, z_n$ be complex numbers. The formula*

$$k(t_1, \ldots, t_m) \;=\; \frac{1}{m'!m''!} \,\mathrm{Sym}\left(f(t_1, \ldots, t_{m'}) g(t_{m'+1}, \ldots, t_m) \prod_{\substack{m' < j \leqslant m \\ 1 \leqslant l \leqslant n'}} \frac{\theta(t_j - z_l + a_l)}{\theta(t_j - z_l - a_l)} \right)$$

correctly defines a linear map $\Phi : f \otimes g \mapsto k = \Phi(f \otimes g)$,

$$\oplus_{m'=0}^{m} F^{m'}_{a_1,\ldots,a_{n'}}(z_1, \ldots, z_{n'}, \lambda) \otimes F^{m''}_{a_{n'+1},\ldots,a_n}(z_{n'+1}, \ldots, z_n, \lambda - 2\nu) \to F^m_{a_1,\ldots,a_n}(z_1, \ldots, z_n, \lambda),$$

where $\nu = a_1 + \cdots + a_{n'} - 2\eta m'$. For generic values of the parameters z_j, λ, the map Φ is an isomorphism. Moreover, Φ is associative in the sense that, for any three functions f, g, h, $\Phi(\Phi(f \otimes g) \otimes h) = \Phi(f \otimes \Phi(g \otimes h))$, whenever defined.

By iterating this construction, we get for all $n \geqslant 1$ a linear map Φ_n, defined recursively by $\Phi_1 = \mathrm{Id}$, $\Phi_n = \Phi(\Phi_{n-1} \otimes \mathrm{Id})$, from

$$\oplus_{m_1 + \cdots + m_n = m} \otimes_{i=1}^{n} F^{m_i}_{a_i}(z_i, \lambda - 2\eta(\mu_1 + \cdots + \mu_{i-1}))$$

to $F^m_{a_1,\ldots,a_n}(z_1, \ldots, z_n, \lambda)$, with $\mu_j = a_j/\eta - 2m_j$, $j = 1, \ldots, n$.

Let $V^*_\Lambda = \oplus_{j=0}^{\infty} \mathbb{C} e^*_j$ be the restricted dual of the module $V_\Lambda = \oplus_{j=0}^{\infty} \mathbb{C} e_j$. It is spanned by the basis (e^*_j) dual to the basis (e_j). We let $\mathfrak{h}$ act on V^*_Λ by $h e^*_j = (\Lambda - 2j) e^*_j$. Then the map that sends e^*_j to ω_j (see (2)) defines an isomorphism of $\mathfrak{h}$-modules

$$\omega(z, \lambda) : V^*_\Lambda \to F_a(z, \lambda), \qquad a = \eta \Lambda.$$

By composing this with the maps Φ of Proposition 2, we obtain homomorphisms (of $\mathfrak{h}$-modules)

$$\omega(z_1, \ldots, z_n, \lambda) : V^*_{\Lambda_1} \otimes \cdots \otimes V^*_{\Lambda_n} \to F_{a_1,\ldots,a_n}(z_1, \ldots, z_n, \lambda)$$

which are isomorphisms for generic values of $z_1, \ldots, z_n, \lambda$. The restriction of the map $\omega(z_1, \ldots, z_n, \lambda)$ to $\mathbb{C} e^*_{m_1} \otimes \cdots \otimes e^*_{m_n}$ is

$$\Phi_n(\omega(z_1, \lambda) e^*_{m_1} \otimes \omega(z_2, \lambda - 2\eta\mu_1) e^*_{m_2} \otimes \cdots \otimes \omega(z_n, \lambda - 2\eta(\mu_1 + \cdots + \mu_{n-1})) e^*_{m_n}),$$

where $\mu_j = \Lambda_j - 2m_j$, $j = 1, \dots, n$. For example, if $n = 2$, then $\omega(z_1, z_2, \lambda)$ sends $e_j^* \otimes e_k^*$ to

$$\frac{1}{j!k!}\,\mathrm{Sym}\left(\omega_j(t_1,\dots,t_j,\lambda;z_1)\omega_k(t_{j+1},\dots,t_{j+k},\lambda-2a_1+4\eta j;z_2)\prod_{i=j+1}^{j+k}\frac{\theta(t_i-z_1+a_1)}{\theta(t_i-z_1-a_1)}\right),$$

where $\{\omega_j(t_1,\dots,t_j,\lambda;z)\}$ is the basis (2) of $F_a(z,\lambda)$.

More generally, we have an explicit formula for the image of $e_{m_1}^* \otimes \cdots \otimes e_{m_n}^*$, which we discuss next.

3.3. **A basis of $F_{a_1,\dots,a_n}(z_1,\dots,z_n)$.** The space V_Λ comes with a basis e_j. Thus we have the natural basis $e_{m_1}^* \otimes \cdots \otimes e_{m_n}^*$ of the tensor product of $V_{\Lambda_i}^*$ in terms of the dual bases of the factors. The map $\omega(z_1,\dots,z_n,\lambda)$ maps, for generic z_i, this basis to a basis of $F_{a_1,\dots,a_n}(z_1,\dots,z_n,\lambda)$, which is an essential part of our formulae for integral representations for solutions of the qKZB equations.

We give here an explicit formula for the basis vectors.

Proposition 3. [FTV] *Let $m \in \mathbb{Z}_{\geqslant 0}$, $\Lambda = (\Lambda_1,\dots,\Lambda_n) \in \mathbb{C}^n$, and let $z = (z_1,\dots,z_n) \in \mathbb{C}^n$ be generic. Set $a_i = \eta\Lambda_i$. Let*

$$u(t_1,\dots,t_m) = \prod_{i<j}\frac{\theta(t_i-t_j+2\eta)}{\theta(t_i-t_j)}$$

Then, for generic $\lambda \in \mathbb{C}$, the functions

$$\omega_{m_1,\dots,m_n}(t_1,\dots,t_m,\lambda;z) \;=\; \omega(z,\lambda)\, e_{m_1}^* \otimes \cdots \otimes e_{m_n}^*$$

labeled by $m_1,\dots,m_n \in \mathbb{Z}$ with $\sum_k m_k = m$ form a basis of $F_a^m(z,\lambda)$ and are given by the explicit formula

$$\omega_{m_1,\dots,m_n}(t_1,\dots,t_m,\lambda;z;\tau) \;=\; u(t_1,\dots,t_m)^{-1}\sum_{I_1,\dots,I_n}\prod_{l=1}^{n}\prod_{i\in I_l}\prod_{k=1}^{l-1}\frac{\theta(t_i-z_k+a_k)}{\theta(t_i-z_k-a_k)} \tag{3}$$
$$\times\prod_{k<l}\prod_{i\in I_k,j\in I_l}\frac{\theta(t_i-t_j+2\eta)}{\theta(t_i-t_j)}\prod_{k=1}^{n}\prod_{j\in I_k}\frac{\theta(\lambda+t_j-z_k-a_k+2\eta m_k-2\eta\sum_{l=1}^{k-1}(\Lambda_l-2m_l))}{\theta(t_j-z_k-a_k)}\,.$$

The summation is over all n-tuples $I_1,\dots,I_n$ of disjoint subsets of $\{1,\dots,m\}$ such that I_j has m_j elements, $1 \leqslant j \leqslant n$.

We shall call the functions $\omega_{m_1,\dots,m_n}(t_1,\dots,t_m,\lambda;z;\tau)$ the weight functions.

3.4. **R-matrices.** Let $a = \eta\Lambda$ and $b = \eta\mathrm{M}$ be complex numbers. Since $F_{ab}(z,w,\lambda)$ coinsides with $F_{ba}(w,z,\lambda)$ by the symmetry of the definition, we obtain a family of isomorphisms between $V_\Lambda^* \otimes V_\mathrm{M}^*$ and $V_\mathrm{M}^* \otimes V_\Lambda^*$. The composition of this family with the flip $P : V_\mathrm{M}^* \otimes V_\Lambda^* \to V_\Lambda^* \otimes V_\mathrm{M}^*$, $Pv \otimes w = w \otimes v$ gives a family of automorphisms of $V_\Lambda^* \otimes V_\mathrm{M}^*$:

Definition: Let z, w, λ be such that $\omega(z,w,\lambda) : V_\Lambda^* \otimes V_\mathrm{M}^* \to F_{ab}(z,w,\lambda)$ is invertible. The *R-matrix* $R_{\Lambda,\mathrm{M}}(z,w,\lambda) \in \mathrm{End}_{\mathfrak{h}}(V_\Lambda \otimes V_\mathrm{M})$ is the dual map to the composition $R_{\Lambda,\mathrm{M}}^*(z,w,\lambda)$:

$$V_\Lambda^* \otimes V_\mathrm{M}^* \xrightarrow{P} V_\mathrm{M}^* \otimes V_\Lambda^* \xrightarrow{\omega(w,z,\lambda)} F_{ab}(z,w,\lambda) \xrightarrow{\omega(z,w,\lambda)^{-1}} V_\Lambda^* \otimes V_\mathrm{M}^*,$$

where we identify canonically $V_\Lambda^* \otimes V_\mathrm{M}^*$ with $(V_\Lambda \otimes V_\mathrm{M})^*$.

Alternatively, the R-matrix $R_{\Lambda,\mathrm{M}}(z,w,\lambda)$ can be thought of as the transition matrix expressing the basis $\tilde\omega_{ij} = \omega(w,z,\lambda)e_j^* \otimes e_i^*$ of the space $F_{ab}(z,w,\lambda)$ in terms of the basis $\omega_{ij} = \omega(z,w,\lambda)e_i^* \otimes e_j^*$: if $R_{\Lambda,\mathrm{M}}(z,w,\lambda)e_i \otimes e_j = \sum_{kl} R_{ij}^{kl} e_k \otimes e_l$, then

$$\tilde\omega_{kl} = \sum_{ij} R_{ij}^{kl}\omega_{ij}. \tag{4}$$

Lemma 4. [FTV]

(i) *$R_{\Lambda,\mathrm{M}}(z,w,\lambda)$ is a meromorphic function of $\Lambda, \mathrm{M}, z, w, \lambda$.*

(ii) *If Λ is generic, then $R_{\Lambda,\Lambda}(z,w,\lambda)$ is regular at $z = w$ and $\lim_{z\to w} R_{\Lambda,\Lambda}(z,w,\lambda) = P$, where P is the flip $u \otimes v \mapsto v \otimes u$.*

(iii) *$R_{\Lambda,\mathrm{M}}(z,w,\lambda)$ depends only on the difference $z - w$.*

Accordingly, we write $R_{\Lambda,\mathrm{M}}(z-w,\lambda)$ instead of $R_{\Lambda,\mathrm{M}}(z,w,\lambda)$ in what follows.

The R-matrices satisfy the dynamical Yang-Baxter equation.

Theorem 5. [FTV] *The matrices $R_{\Lambda,\mathrm{M}}(z,\lambda)$ obey I–III of Section 2.*

Let us now consider the case of positive integer weights. In this case the R-matrices have invariant subspaces. If $\Lambda \in \mathbb{Z}_{\geqslant 0}$ we let SV_Λ be the subspace of V_Λ spanned by $e_{\Lambda+1}, e_{\Lambda+2}, \dots$. The $\Lambda+1$-dimensional quotient V_Λ/SV_Λ will be denoted by L_Λ, and will be often identified with $\oplus_{j=0}^m \mathbb{C}e_j$.

Theorem 6. [FTV] *Let z, η, λ be generic and $\Lambda, \mathrm{M} \in \mathbb{C}$.*

(i) *If $\Lambda \in \mathbb{Z}_{\geqslant 0}$, then $R_{\Lambda,\mathrm{M}}(z,\lambda)$ preserves $SV_\Lambda \otimes V_\mathrm{M}$*
(ii) *If $\mathrm{M} \in \mathbb{Z}_{\geqslant 0}$, then $R_{\Lambda,\mathrm{M}}(z,\lambda)$ preserves $V_\Lambda \otimes SV_\mathrm{M}$*
(iii) *If $\Lambda \in \mathbb{Z}_{\geqslant 0}$ and $\mathrm{M} \in \mathbb{Z}_{\geqslant 0}$, then $R_{\Lambda,\mathrm{M}}(z,\lambda)$ preserves $SV_\Lambda \otimes V_\mathrm{M} + V_\Lambda \otimes SV_\mathrm{M}$.*

In particular, if Λ and/or M are nonnegative integers, then $R_{\Lambda,\mathrm{M}}(z,\lambda)$ induces operators, still denoted by $R_{\Lambda,\mathrm{M}}(z,\lambda)$, on the quotients $L_\Lambda \otimes V_\mathrm{M}$, $V_\Lambda \otimes L_\mathrm{M}$ and/or $L_\Lambda \otimes L_\mathrm{M}$. They obey the dynamical Yang–Baxter equation.

3.5. **Example.** We give an example of computation of matrix elements of the R-matrix $R_{\Lambda,\mathrm{M}}(z-w,\lambda)$, assuming that the parameters are generic.

The R-matrix is calculated as the transition matrix relating two bases of $F_{ab}(z,w,\lambda)$: let

$$\tilde\omega_{ij} = \omega(w,z,\lambda)\, e_j^* \otimes e_i^*, \qquad \omega_{ij} = \omega(z,w,\lambda)\, e_i^* \otimes e_j^*.$$

The matrix elements of R with respect to the basis $e_j \otimes e_k$ are given by $\tilde\omega_{kl} = \sum_{ij} R_{ij}^{kl}\omega_{ij}$. The R-matrix preserves the weight spaces

$$(V_\Lambda \otimes V_\mathrm{M})[\Lambda + \mathrm{M} - 2m] = \oplus_{j=0}^m \mathbb{C}e_j \otimes e_{m-j},$$

and we may consider the problem of computing the matrix elements of the R-matrix separately on each weight space. Without loss of generality we assume that $w = 0$.

Let $m = 0$. Then the weight space is spanned by $e_0 \otimes e_0$ and $\omega_{00} = \tilde\omega_{00} = 1$. Therefore $R_{00}^{00} = 1$.

Let $m = 1$. Then the basis elements are functions of one variable $t = t_1$ and we have (with $a = \eta\Lambda$, $b = \eta\mathrm{M}$)

$$\omega_{01}(t) = \frac{\theta(\lambda + 2\eta + t - 2a - b)\theta(t - z + a)}{\theta(t-b)\theta(t-z-a)}\,, \qquad \omega_{10}(t) = \frac{\theta(\lambda + 2\eta + t - z - a)}{\theta(t-z-a)}\,,$$

and

$$\tilde\omega_{01}(t) = \frac{\theta(\lambda + 2\eta + t - b)}{\theta(t-b)}\,, \qquad \tilde\omega_{10}(t) = \frac{\theta(\lambda + 2\eta + t - z - 2b - a)\theta(t+b)}{\theta(t-z-a)\theta(t-b)}\,.$$

Proposition 7. [FTV]

Let the matrix elements of the R-matrix $R_{\Lambda,\mathrm{M}}(z,\lambda)$ be defined by

$$R_{\Lambda,\mathrm{M}}(z,\lambda)e_i \otimes e_j = \textstyle\sum_{kl} R_{ij}^{kl} e_k \otimes e_l.$$

Then

$$\begin{aligned}
R_{00}^{00} &= 1\,,\\
R_{01}^{01} &= \frac{\theta(z + \eta\Lambda - \eta\mathrm{M})\theta(\lambda + 2\eta)}{\theta(z - \eta\Lambda - \eta\mathrm{M})\theta(\lambda + 2\eta(1-\Lambda))}\,,\\
R_{10}^{01} &= -\frac{\theta(\lambda + 2\eta + z - \eta\Lambda - \eta\mathrm{M})\theta(2\eta\Lambda)}{\theta(z - \eta\Lambda - \eta\mathrm{M})\theta(\lambda + 2\eta(1-\Lambda))}\,,\\
R_{01}^{10} &= -\frac{\theta(\lambda + 2\eta - z - \eta\Lambda - \eta\mathrm{M})\theta(2\eta\mathrm{M})}{\theta(z - \eta\Lambda - \eta\mathrm{M})\theta(\lambda + 2\eta(1-\Lambda))}\,,\\
R_{10}^{10} &= \frac{\theta(z + \eta\mathrm{M} - \eta\Lambda)\theta(\lambda + 2\eta(1-\Lambda-\mathrm{M}))}{\theta(z - \eta\Lambda - \eta\mathrm{M})\theta(\lambda + 2\eta(1-\Lambda))}\,.
\end{aligned}$$

3.6. **Evaluation Verma modules and their tensor products.** Here we explain the relation between the geometric construction of tensor products and R-matrices and the representation theory of $E_{\tau,\eta}(sl_2)$ [FV1].

Recall the definition of a representation of $E_{\tau,\eta}(sl_2)$: let $\mathfrak{h}$ act on $\mathbb{C}^2$ via $h = \mathrm{diag}(1,-1)$. A representation of $E_{\tau,\eta}(sl_2)$ is an $\mathfrak{h}$-module W with diagonalizable action of h and finite-dimensional eigenspaces, together with an operator $L(z,\lambda) \in \mathrm{End}\,(\mathbb{C}^2 \otimes W)$ (the "L-operator"), commuting with $h^{(1)} + h^{(2)}$, and obeying the relations

$$R^{(12)}(z-w,\lambda - 2\eta h^{(3)})\, L^{(13)}(z,\lambda)\, L^{(23)}(w,\lambda - 2\eta h^{(1)}) \;=\; L^{(23)}(w,\lambda)\, L^{(13)}(z,\lambda - 2\eta h^{(2)})\, R^{(12)}(z-w,\lambda)$$

in End $(\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes W)$. The *fundamental R-matrix* $R(z,\lambda) \in \mathrm{End}\,(\mathbb{C}^2 \otimes \mathbb{C}^2)$ is the following solution of the dynamical Yang–Baxter equation: let e_0, e_1 be the standard basis of $\mathbb{C}^2$, then with respect to the basis $e_0 \otimes e_0$, $e_0 \otimes e_1$, $e_1 \otimes e_0$, $e_1 \otimes e_1$ of $\mathbb{C}^2 \otimes \mathbb{C}^2$,

$$R(z,\lambda) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \alpha(z,\lambda) & \beta(z,\lambda) & 0 \\ 0 & \beta(z,-\lambda) & \alpha(z,-\lambda) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

where

$$\alpha(z,\lambda) = \frac{\theta(\lambda+2\eta)\theta(z)}{\theta(\lambda)\theta(z-2\eta)}, \qquad \beta(z,\lambda) = -\frac{\theta(\lambda+z)\theta(2\eta)}{\theta(\lambda)\theta(z-2\eta)}.$$

To discuss representation theory, it is convenient to think of $L(z,\lambda) \in \mathrm{End}\,(\mathbb{C}^2 \otimes W)$ as a two by two matrix with entries $a(z,\lambda)$, $b(z,\lambda)$, $c(z,\lambda)$, $d(z,\lambda)$ in End (W). In [FV1] we wrote explicitly the relations that these four operators must satisfy, and defined a class of representations, the evaluation Verma modules, by giving explicitly the action of these four operators on basis vectors. These formulae can be obtained from the geometric construction.

Theorem 8. [FTV] *Let us identify the two-dimensional space $L_1 = V_{\Lambda=1}/SV_{\Lambda=1}$ with $\mathbb{C}^2$ via the basis e_0, e_1. Then the R-matrix $R_{1,1}(z,\lambda) \in \mathrm{End}\,(L_1 \otimes L_1)$ coincides with the fundamental R-matrix.*

Corollary 9. *For any $w, \mathrm{M} \in \mathbb{C}$, the $\mathfrak{h}$ -module V_{M} together with the operator $L(z,\lambda) = R_{1,\mathrm{M}}(z-w,\lambda) \in \mathrm{End}\,(L_1 \otimes V_{\mathrm{M}})$ defines a representation of $E_{\tau,\eta}(sl_2)$.*

This representation is called in [FV1] the evaluation Verma module with evaluation point w and highest weight M. It is denoted by $V_{\mathrm{M}}(w)$. The matrix elements of $L(z,\lambda)$ are given explicitly in Theorem 3 of [FV1] in terms of the action of $a(z,\lambda), \dots, d(z,\lambda)$. In the notation of Proposition 7, this result amounts to the following formulae for the matrix elements R_{ij}^{kl} of $R_{1,\Lambda}(z,\lambda) \in \mathrm{End}\,(L_1 \otimes V_\Lambda)$.

$$\begin{aligned}
R_{0k}^{0k} &= \frac{\theta(z-(\Lambda+1-2k)\eta)}{\theta(z-(\Lambda+1)\eta)}\frac{\theta(\lambda+2k\eta)}{\theta(\lambda)}, \\
R_{1k}^{0,k+1} &= -\frac{\theta(\lambda+z-(\Lambda-1-2k)\eta)}{\theta(z-(\Lambda+1)\eta)}\frac{\theta(2\eta)}{\theta(\lambda)}, \\
R_{0k}^{1,k-1} &= -\frac{\theta(\lambda-z-(\Lambda+1-2k)\eta)}{\theta(z-(\Lambda+1)\eta)}\frac{\theta(2(\Lambda+1-k)\eta)}{\theta(\lambda)}\frac{\theta(2k\eta)}{\theta(2\eta)}, \\
R_{1k}^{1k} &= \frac{\theta(z-(-\Lambda+1+2k)\eta)}{\theta(z-(\Lambda+1)\eta)}\frac{\theta(\lambda-2(\Lambda-k)\eta)}{\theta(\lambda)}.
\end{aligned}$$

Moreover, the tensor product construction of 3.2 is related to the tensor product of representations of the elliptic quantum group. Recall that if W_1, W_2 are representations of the elliptic quantum group with L-operators $L_1(z,\lambda)$, $L_2(z,\lambda)$, then their tensor product $W = W_1 \otimes W_2$ with L-operator

$$L(z,\lambda) = L_1(z,\lambda-2\eta h^{(3)})^{(12)}L_2(z,\lambda)^{(13)} \in \mathrm{End}\,(\mathbb{C}^2 \otimes W)$$

is also a representation of the elliptic quantum group.

Theorem 10. [FTV]
Let $\Lambda_1, \dots, \Lambda_n \in \mathbb{C}$ and $z_1, \dots, z_n$ be generic complex numbers. Let $V = V_{\Lambda_1} \otimes \cdots \otimes V_{\Lambda_n}$ and $L(z,\lambda) \in \mathrm{End}\,(V_{\Lambda=1} \otimes V)$ be defined by the relation

$$\omega(z, z_1, \dots, z_n, \lambda)L(z,\lambda)^* = \omega(z_1, \dots, z_n, z, \lambda)P$$

in End $((V_1 \otimes V)^*) =$ End $(V_1^* \otimes V^*)$, *where $Pv_1 \otimes v = v \otimes v_1$, if $v_1 \in V_1^*$, $v \in V^*$. Then $L(z,\lambda)$ is well-defined as an endomorphism of the quotient $L_1 \otimes V = \mathbb{C}^2 \otimes V$, and defines a structure of a representation of $E_{\tau,\eta}(sl_2)$ on V. This representation is isomorphic to the tensor product of evaluation Verma modules*

$$V_{\Lambda_n}(z_n) \otimes \cdots \otimes V_{\Lambda_1}(z_1),$$

with the isomorphism $u_1 \otimes \cdots \otimes u_n \mapsto u_n \otimes \cdots \otimes u_1$.

Finally, the dynamical Yang–Baxter equation in $L_1 \otimes V_\Lambda \otimes V_{\mathrm{M}}$ can be stated as saying that $R_{\Lambda,\mathrm{M}}(z-w,\lambda)P$ is an isomorphism from $V_{\mathrm{M}}(w) \otimes V_\Lambda(z)$ to $V_\Lambda(z) \otimes V_{\mathrm{M}}(w)$, see [FV1]. Therefore, by uniqueness, the R-matrices constructed in this section coincide with the solutions of the dynamical Yang–Baxter equation described in Section 13 of [FV1].

4. Transformation of qKZB with respect to shifts of z_k by τ and 1

4.1. **Transformation of weight functions.** Let $\omega_{m_1,\dots,m_n}(t,\lambda;z_1,\dots,z_n;\tau)$ be the weight functions defined by (3). The next Proposition describes the transformation properties of the weight functions with respect to shifts by τ of variables $z_1,\dots,z_n$.

Proposition 11. *For any k, we have*

$$\omega_{m_1,\dots,m_n}(t,\lambda;z_1,\dots,z_k+\tau,\dots,z_n;\tau)=e^{b_k}\omega_{m_1,\dots,m_n}(t,\lambda;z_1,\dots,z_k,\dots,z_n;\tau)$$

where

$$b_k=2\pi i m_k(\lambda+2\eta m_k-2\sum_{l=1}^{k-1}(a_l-2\eta m_l))+4\pi i a_k\sum_{l=k+1}^{n}m_l.$$

The Proposition follows from transformation properties of $\theta(t)$.

We reformulate the Proposition. Introduce a function

$$\alpha(\lambda)\;=\;\exp(-\pi i\lambda^2/4\eta)\,. \tag{5}$$

Use the notation $h^{(j)}=a_j/\eta-2m_j$. Set

$$D_k(\lambda)\;=\;\frac{\alpha(\lambda\,-\,2\eta\sum_{l=1}^{k}h^{(l)})}{\alpha(\lambda\,-\,2\eta\sum_{l=1}^{k-1}h^{(l)})}\,e^{\pi i\eta\Lambda_k(\sum_{l=1}^{k-1}\Lambda_l-\sum_{l=k+1}^{n}\Lambda_l)}. \tag{6}$$

Then

$$e^{b_k}\;=\;D_k^{-1}\,e^{\pi i\Lambda_k(\lambda+2\eta m)}\,e^{-\pi i\eta\Lambda_k\sum_{l=1}^{n}h^{(l)}}\,, \tag{7}$$

where $m=m_1+\ldots+m_n$.

4.2. **Transformations of R-matrices.** Consider the tensor product of evaluation Verma modules over the elliptic quantum group $E_{\tau,\eta}(sl_2)$, $V_\Lambda(z)\otimes V_{\mathrm{M}}(w)$, and its R-matrix $R_{\Lambda,\mathrm{M}}(z-w,\lambda)\in\mathrm{End}\,(V_\Lambda\otimes V_{\mathrm{M}})$ defined in Section 3.6. Let $h^{(j)}$ denote the operator h acting in the j-th factor of $V_\Lambda(z)\otimes V_{\mathrm{M}}(w)$.

Proposition 12.

$$R_{\Lambda,\mathrm{M}}(z+\tau,\lambda)\;=\;e^{-2\pi i\Lambda\mathrm{M}}\,\frac{\alpha(\lambda\,-\,2\eta h^{(2)})}{\alpha(\lambda\,-\,2\eta\,(h^{(1)}+h^{(2)}))}\,R_{\Lambda,\mathrm{M}}(z,\lambda)\,\frac{\alpha(\lambda\,-\,2\eta h^{(1)})}{\alpha(\lambda)}\,. \tag{8}$$

Proof: Use formula (4),

$$\tilde\omega_{kl}(z,w)\;=\;\sum_{ij}R_{ij}^{kl}(z-w,\lambda)\,\omega_{ij}(z,w).$$

Then

$$\tilde\omega_{kl}(z+\tau,w)\;=\;\sum_{ij}R_{ij}^{kl}(z-w+\tau,\lambda)\,\omega_{ij}(z+\tau,w).$$

We have $\tilde\omega_{kl}(z+\tau,w)=e^{\tilde b_1(kl)}\tilde\omega_{kl}(z,w)$ and $\omega_{ij}(z+\tau,w)=e^{b_1(ij)}\omega_{ij}(z,w)$ where $e^{\tilde b_1}$ and e^{b_1} are given by (7). Hence

$$R_{ij}^{kl}(z-w+\tau,\lambda)\;=\;e^{\tilde b_1(kl)}\,R_{ij}^{kl}(z-w,\lambda)\,e^{-b_1(ij)}.$$

This proves the Proposition. □

Proposition 13.

$$R_{\Lambda,\mathrm{M}}(z-w,\lambda+\tau)\;=\;e^{\pi i(h^{(1)}(-z-\eta\mathrm{M})+h^{(2)}(-w+\eta\Lambda))}\,R_{\Lambda,\mathrm{M}}(z-w,\lambda)\,e^{\pi i(h^{(1)}(z-\eta\mathrm{M})+h^{(2)}(w+\eta\Lambda))}\,.$$

The proof is analogous to the proof of Proposition 12. □

Proposition 14.

$$R_{\Lambda,\mathrm{M}}(z+1,\lambda,\tau)\;=\;R_{\Lambda,\mathrm{M}}(z,\lambda+1,\tau)\;=\;R_{\Lambda,\mathrm{M}}(z,\lambda,\tau+1)\;=\;R_{\Lambda,\mathrm{M}}(z,\lambda,\tau)\,.$$

The proof easily follows from the formulae $\theta(t+1,\tau)=-\theta(t,\tau)$ and $\theta(t,\tau+1)=e^{\pi i/4}\theta(t,\tau)$.

4.3. **Transformation of qKZB equations with respect to shifts $z_k \to z_k + \tau$.** Consider the qKZB equations defined by (1),

$$\Psi(z_1, \dots, z_j + p, \dots, z_n, \lambda) \,=\, K_j(z_1, \dots, z_n, \lambda)\, \Psi(z_1, \dots, z_n, \lambda)\,, \qquad j = 1, \dots, n,$$

where $\Psi(z_1, \dots, z_n, \lambda)$ is a function with values in $V = V_{\Lambda_1} \otimes \dots \otimes V_{\Lambda_n}$ and $K_j(z_1, \dots, z_n, \lambda) \in \mathrm{End}\,(V)$.

For any $k = 1, \dots, n$, introduce a linear operator $B_k(z_1, \dots, z_n, \lambda) \in \mathrm{End}\,(V[0])$ by

$$B_k(z_1, \dots, z_n, \lambda) \,=\, e^{2\pi i\eta \sum_{l,\, l\neq k} (z_l - z_k)\Lambda_l\Lambda_k/p}\ \frac{\alpha(\lambda \,-\, 2\eta \sum_{l=1}^{k} h^{(l)})}{\alpha(\lambda \,-\, 2\eta \sum_{l=1}^{k-1} h^{(l)})}\, e^{\pi i\eta\Lambda_k(\sum_{l=1}^{k-1}\Lambda_l - \sum_{l=k+1}^{n}\Lambda_l)}. \tag{9}$$

Theorem 15. *For any j and k, we have*

$$\begin{aligned} &K_j(z_1, \dots, z_k + \tau, \dots, z_n, \lambda) \,= \\ &B_k(z_1, \dots, z_j + p, \dots, z_n, \lambda)^{-1}\, K_j(z_1, \dots, z_n, \lambda)\, B_k(z_1, \dots, z_j, \dots, z_n, \lambda)\,. \end{aligned} \tag{10}$$

Notice that the last exponential in (9) is not essential for the Theorem and is introduced for later purposes.

Proof: There are three cases: $k > j$, $k = j$, $k < j$. We prove the Theorem for $k > j$. The other two cases are proved similarly.

$$\begin{aligned} &K_j(z_1, \dots, z_k + \tau, \dots, z_n, \lambda) \,=\, R_{j,j-1}(z_j - z_{j-1} + p) \cdots R_{j,1}(z_j - z_1 + p)\Gamma_j R_{j,n}(z_j - z_n) \cdots \\ &R_{j,k}(z_j - z_k \,-\, \tau) \cdots R_{j,j+1}(z_j - z_{j+1}) \,=\, R_{j,j-1}(z_j - z_{j-1} + p) \cdots R_{j,1}(z_j - z_1 + p)\Gamma_j R_{j,n}(z_j - z_n) \cdots \\ &e^{2\pi i\eta\Lambda_j\Lambda_k} \frac{\alpha(\lambda \,-\, 2\eta \sum_{l=1}^{k} h^{(l)})}{\alpha(\lambda \,-\, 2\eta \sum_{l=1,\, l\neq j}^{k} h^{(l)})} R_{j,k}(z_j - z_k) \frac{\alpha(\lambda \,-\, 2\eta \sum_{l=1,\, l\neq j}^{k-1} h^{(l)})}{\alpha(\lambda \,-\, 2\eta \sum_{l=1}^{k-1} h^{(l)})} \cdots R_{j,j+1}(z_j - z_{j+1}) \,= \\ &e^{2\pi i\eta\Lambda_j\Lambda_k} R_{j,j-1}(z_j - z_{j-1} + p) \cdots R_{j,1}(z_j - z_1 + p)\Gamma_j R_{j,n}(z_j - z_n) \cdots \frac{\alpha(\lambda \,-\, 2\eta \sum_{l=1,\, l\neq j}^{k-1} h^{(l)})}{\alpha(\lambda \,-\, 2\eta \sum_{l=1,\, l\neq j}^{k} h^{(l)})} \\ &R_{j,k}(z_j - z_k) \frac{\alpha(\lambda \,-\, 2\eta \sum_{l=1}^{k} h^{(l)})}{\alpha(\lambda \,-\, 2\eta \sum_{l=1}^{k-1} h^{(l)})} \cdots R_{j,j+1}(z_j - z_{j+1}) \,=\, e^{2\pi i\eta\Lambda_j\Lambda_k} R_{j,j-1}(z_j - z_{j-1} + p) \cdots \\ &R_{j,1}(z_j - z_1 + p)\Gamma_j \frac{\alpha(\lambda \,-\, 2\eta \sum_{l=1,\, l\neq j}^{k-1} h^{(l)})}{\alpha(\lambda \,-\, 2\eta \sum_{l=1,\, l\neq j}^{k} h^{(l)})} R_{j,n}(z_j - z_n) \cdots R_{j,k}(z_j - z_k) \cdots R_{j,j+1}(z_j - z_{j+1}) \\ &\frac{\alpha(\lambda \,-\, 2\eta \sum_{l=1}^{k} h^{(l)})}{\alpha(\lambda \,-\, 2\eta \sum_{l=1}^{k-1} h^{(l)})} \,=\, e^{2\pi i\eta\Lambda_j\Lambda_k} \frac{\alpha(\lambda \,-\, 2\eta \sum_{l=1}^{k-1} h^{(l)})}{\alpha(\lambda \,-\, 2\eta \sum_{l=1}^{k} h^{(l)})} R_{j,j-1}(z_j - z_{j-1} + p) \cdots \\ &R_{j,1}(z_j - z_1 + p)\Gamma_j R_{j,n}(z_j - z_n) \cdots R_{j,k}(z_j - z_k) \cdots R_{j,j+1}(z_j - z_{j+1}) \frac{\alpha(\lambda \,-\, 2\eta \sum_{l=1}^{k} h^{(l)})}{\alpha(\lambda \,-\, 2\eta \sum_{l=1}^{k-1} h^{(l)})} \,= \\ &e^{2\pi i\eta\Lambda_j\Lambda_k} \frac{\alpha(\lambda \,-\, 2\eta \sum_{l=1}^{k-1} h^{(l)})}{\alpha(\lambda \,-\, 2\eta \sum_{l=1}^{k} h^{(l)})} K_j(z_1, \dots, z_n, \lambda) \frac{\alpha(\lambda \,-\, 2\eta \sum_{l=1}^{k} h^{(l)})}{\alpha(\lambda \,-\, 2\eta \sum_{l=1}^{k-1} h^{(l)})}. \end{aligned}$$

This proves the Theorem for $k > j$. □

Theorem 16. *Let $\Psi(z_1, \dots, z_n, \lambda)$ be a solution of the qKZB equations (1). Then for any k, the function $B_k(z_1, \dots, z_n, \lambda)\Psi(z_1, \dots, z_k + \tau, \dots, z_n, \lambda)$ is a new solution of the same equations.*

Proof: For any j, we have

$$\begin{aligned} &B_k(z_1, \dots, z_j + p, \dots, z_n, \lambda)\, \Psi(z_1, \dots, z_k + \tau, \dots, z_j + p, \dots, z_n, \lambda) \,= \\ &B_k(z_1, \dots, z_j + p, \dots, z_n, \lambda)\, K_j(z_1, \dots, z_k + \tau, \dots, z_n, \lambda)\, \Psi(z_1, \dots, z_k + \tau, \dots, z_j, \dots, z_n, \lambda) \,= \\ &B_k(z_1, \dots, z_j + p, \dots, z_n, \lambda)\, B_k(z_1, \dots, z_j + p, \dots, z_n, \lambda)^{-1}\, K_j(z_1, \dots, z_n, \lambda)\, B_k(z_1, \dots, z_n, \lambda) \\ &\Psi(z_1, \dots, z_k + \tau, \dots, z_n, \lambda) \,=\, K_j(z_1, \dots, z_n, \lambda)\, B_k(z_1, \dots, z_n, \lambda)\, \Psi(z_1, \dots, z_k + \tau, z_n, \lambda)\,. \end{aligned}$$

□

4.4. **Transformation of qKZB equations with respect to shifts $z_k \to z_k + 1$.**

Proposition 17. *For any j, we have*

$$K_j(z_1, \dots, z_k + 1, \dots, z_n,\, \lambda) \,=\, K_j(z_1, \dots, z_k, \dots, z_n,\, \lambda)\,.$$

The Proposition follows from Proposition 14.

Corollary 18. *Let $\Psi(z_1,\dots,z_n,\lambda)$ be a solution of the qKZB equations (1). Then for any k, the function $\Psi(z_1,\dots,z_k+1,\dots,z_n,\lambda)$ is a new solution of the same equations.*

4.5. **The qKZB equations as equations on a torus.** Propositions 16 and 18 mean that we can define a vector bundle over the torus $\mathbb{C}^n\,/\,\mathbb{Z}^n+\tau\mathbb{Z}^n$ whose fiber is the space of functions of λ with values in $V[0]$. The identification of points of the base, $(z_1,\dots,z_n)\to(z_1,\dots,z_k+\tau,\dots,z_n)$, corresponds to the identification of the fibers defined by $v\to B_k(z_1,\dots,z_n)v$. The identification of points of the base, $(z_1,\dots,z_n)\to(z_1,\dots,z_k+1,\dots,z_n)$, corresponds to the identification of the fibers defined by $v\to v$. The identifications commute with the qKZB equations. Hence the qKZB equations induce a system of difference equations on the torus with values in this bundle (a flat discrete connection). A solution of the initial qKZB equations defines a multivalued solution of the difference equations on the torus. Now we can ask a question about the monodromy of solutions of the equations on the torus. We shall address this problem in Section 7

5. Formal solutions of the qKZB equations

In this section we fix $\tau,\eta,p,\Lambda_1,\dots,\Lambda_n$, and set $a_i=\eta\Lambda_i$.

5.1. **Formal integral solutions.** By a formal Jackson integral solution of the qKZB equations we mean an expression

$$\Psi(z_1,\dots,z_n,\lambda)=\int f(z_1,\dots,z_n,t_1,\dots,t_m,\lambda)Dt_1\cdots Dt_m,$$

where f takes its values in $V[0]$, which obeys the qKZB equations (1) if we formally use the rule that the "integral" $\int$ is invariant under translations of the variables t_i by p. In other words, $f(z_1,\dots,z_n,t_1,\dots,t_m,\lambda)$ obeys the qKZB equations in the variables z_i up to terms of the form $g(\dots,t_i+p,\dots)-g(\dots,t_i,\dots)$.

Definition: A function $\Phi_a(t)$ depending on a complex parameter a, such that

$$\Phi_a(t+p)=\frac{\theta(t+a)}{\theta(t-a)}\Phi_a(t)$$

is called a (one-variable) phase function.

We assume that p has positive imaginary part, and set $r=e^{2\pi ip}$, $q=e^{2\pi i\tau}$. Then the convergent infinite product

$$\Omega_a(t):=\Omega_a(t,\tau,p)=\prod_{j=0}^{\infty}\prod_{k=0}^{\infty}\frac{(1-r^jq^ke^{2\pi i(t-a)})(1-r^{j+1}q^{k+1}e^{-2\pi i(t+a)})}{(1-r^jq^ke^{2\pi i(t+a)})(1-r^{j+1}q^{k+1}e^{-2\pi i(t-a)})}\,, \tag{11}$$

defines a phase function

$$\Phi_a(t)=e^{-2\pi iat/p}\Omega_a(t)$$

and any other phase function is obtained from this one by multiplication by a p-periodic function.

Given a one-variable phase function $\Phi_a(t)$, we define with our data an m-variable phase function

$$\Phi(t_1,\dots,t_m,z_1,\dots,z_n)=\prod_{j=1}^{m}\prod_{l=1}^{n}\Phi_{a_l}(t_j-z_l)\prod_{1\leqslant i<j\leqslant m}\Phi_{-2\eta}(t_i-t_j). \tag{12}$$

Theorem 19. [FTV]. *Let $\Phi_a(t)$ be a phase function, and let Φ be the corresponding m-variable phase function (12). For any entire function ξ of one variable, let*

$$\psi^\xi(t,z,\lambda)=\xi(p\lambda-\sum_{l=1}^{n}2a_lz_l+4\eta\sum_{j=1}^{m}t_j))\sum_{j_1+\dots+j_n=m}\omega_{j_1,\dots,j_n}(t_1,\dots,t_m,\lambda)\,e_{j_1}\otimes\dots\otimes e_{j_n}.$$

Then

$$\Psi(z_1,\dots,z_n,\lambda)=\int\Phi(t_1,\dots,t_m,z_1,\dots,z_n)\psi^\xi(t_1,\dots,t_m,z_1,\dots,z_n,\lambda)Dt_1\cdots Dt_m$$

is a formal Jackson integral solution of the qKZB equations.

The proof of the theorem is based on the transformation properties of the m-variable phase function $\Phi(t,z)$ with respect to shifts $t_j \mapsto t_j + p$, $z_l \mapsto z_l + p$ and does not use the explicit form of the function $\Phi(t,z)$, see [FTV].

To obtain solutions from formal Jackson integral solutions, we need to find *cycles*, linear forms on the space of functions of $t_1, \dots, t_m$ that are invariant under translations $t_i \mapsto t_i + p$. To this end we need a stronger version of the preceding theorem, Theorem 21 below gives us a space of functions on which our cycles should be defined.

Let Φ be the phase function (12) and let $a = (a_1, \dots, a_n)$, $z = (z_1, \dots, z_n)$. We assume, as usual, that $\sum a_i = 2\eta m$, $m \in \mathbb{Z}_{\geqslant 0}$. For any entire function ξ of one variable, let $E_a^0(z;\xi)$, be the space spanned by the functions of $t \in \mathbb{C}^m$ of the form

$$\Phi(t,z)\xi(p\lambda - \sum_{k=1}^n 2a_k z_k + 4\eta \sum_{j=1}^m t_j) f(t,z),$$

where $f(t,z)$, viewed as a function of $t = (t_1, \dots, t_m)$ belongs to $\tilde{F}_a^m(z,\lambda)$ (see 3.1) for some λ. All components of our integrand belong to this space.

Let $E_a(z;\xi)$, the space of cocycles, be the space spanned by functions of the form $g(t + p\alpha)$, where $g \in E_a^0(z;\xi)$ and $\alpha \in \mathbb{Z}^m$. By construction, $E_a(z;\xi)$ is invariant under translations of the arguments t_i by p. We define the space of coboundaries $DE_a(z;\xi)$ to be the subspace of $E_a(z;\xi)$ spanned by functions of the form $f(\dots, t_j + p, \dots) - f(\dots, t_j, \dots)$, $f \in E_a(z;\xi)$.

Proposition 20. [FTV]. *$E_a(z_1, \dots, z_n) = E_a(z_1, \dots, z_j + p, \dots, z_n)$ for $j = 1, \dots, n$*

Moreover, we have the following result.

Theorem 21. [FTV]. *Let us write the qKZB equations as $\Psi(\dots, z_j + p, \dots) = K_j(z)\Psi(z)$. Then, for any entire function ξ, the integrand $\Phi(t,z)\psi^\xi(t,z)$ of Theorem 19, viewed as a function of $t \in \mathbb{C}^m$, belongs to $E_a(z;\xi) \otimes V[0]$ for all $z \in \mathbb{C}^n$. It obeys the equations*

$$\Psi(t, \dots, z_j + p, \dots) = K_j(z)\Psi(t,z) \mod DE_a(z;\xi) \otimes V[0], \qquad j = 1, \dots, n.$$

In other words, $\Psi(z,t)$ solves the qKZB equations in the cohomology $(E_a(z;\xi)/DE_a(z;\xi)) \otimes V[0]$.

To obtain solutions from these formal solutions, one should find horizontal families of cycles, i.e., linear functions $\gamma(z)$ on $E_a(z;\xi)$ vanishing on $DE_a(z;\xi)$, and such that $\gamma(z + p\alpha) = \gamma(z)$ for all $\alpha \in \mathbb{Z}^n$. This problem will be addressed in the next section.

5.2. **New formal solutions.** Let $a = (a_1, \dots, a_n)$, $z = (z_1, \dots, z_n)$, $t = (t_1, \dots, t_m)$. Assume that $\sum a_i = 2\eta m$ and the step p of the qKZB equations has a positive imaginary part. In this section we shall construct a finite set of new formal integral solutions to the qKZB equations (1). The formal solutions are labelled by an index $M = (m_1, \dots, m_n)$ where non-negative integers m_i satisfy $m_1 + \dots + m_n = m$. Each of the formal solutions will depend on a complex parameter μ.

We shall use the notation $h^{(j)} = a_j/\eta - 2m_j$. According to our assumptions we have $h^{(1)} + \dots + h^{(n)} = 0$.

Introduce a new m-variable phase function

$$\Omega(t_1, \dots, t_m, z_1, \dots, z_n, \tau, p) = \prod_{j=1}^m \prod_{l=1}^n \Omega_{a_l}(t_j - z_l) \prod_{1 \leqslant i < j \leqslant m} \Omega_{-2\eta}(t_i - t_j). \tag{13}$$

Introduce a function

$$D_M(\mu, z_1, \dots, z_n) = \prod_{k=1}^n D_k(\mu)^{z_k/p} \tag{14}$$

where D_k are defined in (6).

Let $\omega_M(t, \mu, z; p)$ be the function defined in (3). Notice that the parameter τ of the theta functions in (3) is replaced here by p and λ is replaced by μ. It follows from (7) that the function $\omega_M(t, \mu, z; p) D_M(\mu, z)$ has the following transformation properties,

$$\begin{aligned} &\omega_M(t, \mu, \dots, z_j + p, \dots; p)\, D_M(\mu, \dots, z_j + p, \dots) \\ &= e^{2\pi i(\frac{\mu}{2\eta} + m)a_j}\, \omega_M(t, \mu, \dots, z_j, \dots; p)\, D_M(\mu, \dots, z_j, \dots) \end{aligned}$$

for $j = 1, \dots, n$.

For any entire function ξ of one variable, let $E_a^0(z;\xi;M)$, be the space spanned by the functions of $t \in \mathbb{C}^m$ of the form

$$\xi(p\lambda - \sum_{k=1}^n 2a_k z_k + 4\eta \sum_{j=1}^m t_j)\Omega(t,z)\omega_M(t, \mu, z; p)\, D_M(\mu, z) f(t,z),$$

where $f(t,z)$, viewed as a function of $t=(t_1,\dots,t_m)$ belongs to $\tilde F^m_a(z,\lambda)$ (see 3.1) for some λ.

Let $E_a(z;\xi;M)$, the space of cocycles, be the space spanned by functions of the form $g(t+p\alpha)$, where $g\in E^0_a(z;\xi;M)$ and $\alpha\in\mathbb{Z}^m$. By construction, $E_a(z;\xi;M)$ is invariant under translations of the arguments t_i by p. We define the space of coboundaries $DE_a(z;\xi)$ to be the subspace of $E_a(z;\xi;M)$ spanned by functions of the form $f(\dots,t_j+p,\dots)-f(\dots,t_j,\dots)$, $f\in E_a(z;\xi)$.

Proposition 22. *$E_a(z_1,\dots,z_n;\xi;M)=E_a(z_1,\dots,z_j+p,\dots,z_n;\xi;M)$ for $j=1,\dots,n$*

The proof of this Proposition is the same as the proof of Proposition 20, see [FTV].

Theorem 23. *Let us write the qKZB equations as $\Psi(\dots,z_j+p,\dots)=K_j(z)\Psi(z)$. Then, for all entire functions ξ, the integrand*

$$e^{-\pi i\frac{\mu\lambda}{2\eta}}\,\Omega(t,z)\,\omega_M(t,\mu,z;p)\,D_M(\mu,z)\psi^\xi(t,z), \tag{15}$$

viewed as a function of $t\in\mathbb{C}^m$, belongs to $E_a(z;\xi;M)\otimes V[0]$ for all $z\in\mathbb{C}^n$. It obeys the equations

$$\Psi(t,\dots,z_j+p,\dots)=K_j(z)\Psi(t,z)\mod DE_a(z;\xi;M)\otimes V[0],\qquad j=1,\dots,n.$$

In other words, $\Psi(z,t)$ solves the qKZB equations in the cohomology $(E_a(z;\xi;M)/DE_a(z;\xi;M))\otimes V[0]$.

Consider a square matrix I of size dim $V[0]$ with entries

$$I_{L,M}\;=\;e^{-\pi i\frac{\mu\lambda}{2\eta}}\,\xi(\tau\mu+p\lambda-\sum_{k=1}^n 2a_kz_k+4\eta\sum_{j=1}^m t_j)\Omega(t,z)\,\omega_L(t,\lambda,z;\tau)\,\omega_M(t,\mu,z;p)\,e_L\otimes D_M(\mu,z)e_M,$$

where $L=(l_1,\dots,l_n)$, $M=(m_1,\dots,m_n)$, and $e_L=e_{l_1}\otimes\dots\otimes e_{l_n}$, $e_M=e_{m_1}\otimes\dots\otimes e_{m_n}$ are basis elements of $V[0]$. The entries of the matrix are functions of τ,p,λ,μ,t,z. If we ignore the factor D_M, the matrix will be invariant under exchange of p,λ,L and τ,μ,M.

Corollary 24. *For every M the corresponding column $I_M=(I_{L,M})$ of the matrix is a formal solution of the qKZB equations.*

Proof: Theorem 23 follows from the proof of Theorem 21. In fact, according to Theorem 21 we know that $\Phi(t,z)\psi^\xi(t,z,\lambda)$ is a formal solution. Now we can write

$$\begin{aligned}
&\Phi(t,z)\psi^\xi(t,z,\lambda)=e^{-\pi i\frac{\mu\lambda}{2\eta}}\,e^{\pi i\frac{\mu\lambda}{2\eta}}\,\Phi(t,z)\psi^\xi(t,z,\lambda)=\\
&=e^{-\pi i\frac{\mu\lambda}{2\eta}}\,\Omega(t,z)\psi^\xi(t,z,\lambda)\,e^{\pi i\frac{\mu\lambda}{2\eta}}\,e^{\frac{2\pi i}{p}(m\sum_{l=1}^n a_lz_l+\sum_{j=1}^m(2\eta-4\eta j)t_j)}=\\
&=e^{-\pi i\frac{\mu\lambda}{2\eta}}\,\Omega(t,z)\psi^\xi(t,z,\lambda)\,e^{\pi i\frac{\mu}{2\eta p}(p\lambda-2\sum_{l=1}^n a_lz_l+4\eta\sum_{j=1}^m t_j)}\times\\
&\quad\times e^{\pi i\frac{\mu}{2\eta p}(2\sum_{l=1}^n a_lz_l-4\eta\sum_{j=1}^m t_j)}\,e^{\frac{2\pi i}{p}(m\sum_{l=1}^n a_lz_l+\sum_{j=1}^m(2\eta-4\eta j)t_j)}=\\
&=e^{-\pi i\frac{\mu\lambda}{2\eta}}\,\Omega(t,z)\psi^\xi(t,z,\lambda)\,e^{\frac{2\pi i}{p}(\frac{\mu}{2\eta}+m)\sum_{l=1}^n a_lz_l}\,e^{\frac{-2\pi i}{p}\sum_{j=1}^m(\mu+4\eta j-2\eta)t_j}\times\\
&\quad\times e^{\pi i\frac{\mu}{2\eta p}(p\lambda-2\sum_{l=1}^n a_lz_l+4\eta\sum_{j=1}^m t_j)}\,.
\end{aligned}$$

The last factor can be included into the entire function ξ. Thus we proved that

$$e^{-\pi i\frac{\mu\lambda}{2\eta}}\,\Omega(t,z)\psi^\xi(t,z,\lambda)\,e^{\frac{2\pi i}{p}(\frac{\mu}{2\eta}+m)\sum_{l=1}^n a_lz_l}\,e^{\frac{-2\pi i}{p}\sum_{j=1}^m(\mu+4\eta j-2\eta)t_j} \tag{16}$$

is a formal solution depending on an additional parameter μ.

Now choose $M=(m_1,\dots,m_n)$ as in Theorem 23. The function $\omega_M(t,\mu,z;p)\,D_M(\mu,z)$ has the same transformation properties under the shifts $t_j\mapsto t_j+p$ and $z_l\mapsto z_l+p$ as the product of the last two factors in (16). Hence

$$e^{-\pi i\frac{\mu\lambda}{2\eta}}\,\Omega(t,z)\psi^\xi(t,z,\lambda)\omega_M(t,\mu,z;p)\,D_M(\mu,z)$$

is a formal integral solution. Theorem 23 is proved. $\square$

6. Integration

6.1. **Poles of the new formal solutions.** Let $t=(t_1,\dots,t_m)$, $z=(z_1,\dots,z_n)$. Let $\Omega(t,z,\tau,p,\eta)$ be the m-variable phase function introduced in (13). Let $\omega_{l_1,\dots,l_n}(t,\lambda,z,\tau,\eta)$ and $\omega_{m_1,\dots,m_n}(t,\mu,z,p,\eta)$ be the functions introduced in (3), here $l_1+\dots+l_n=m$ and $m_1+\dots+m_n=m$.

Proposition 25. *The poles of the function*

$$\Omega(t,z,\tau,p,\eta)\,\omega_{l_1,\dots,l_n}(t,\lambda,z,\tau,\eta)\,\omega_{m_1,\dots,m_n}(t,\mu,z,p,\eta) \tag{17}$$

are of first order and lie at the hyperplanes given by equations

$$t_i - z_k = -a_k - rp - s\tau + l, \qquad t_i - z_k = a_k + rp + s\tau + l, \tag{18}$$

where $i = 1,\dots,m$, $k = 1,\dots,n$ and $r,s \in \mathbb{Z}_{\geqslant 0}$, $l \in \mathbb{Z}$, and

$$t_i = t_j + 2\eta - rp - s\tau + l, \qquad t_i = t_j - 2\eta + rp + s\tau + l, \tag{19}$$

where $1 \leqslant i < j \leqslant m$ and $r,s \in \mathbb{Z}_{\geqslant 0}$, $l \in \mathbb{Z}$.

Proof: The poles of the function $\Omega_a(t,\tau,p)$ introduced in (11) are of first order and are given by

$$t = -a - rp - s\tau + l, \qquad t = a + (r+1)p + (s+1)\tau + l,$$

and the zeros are

$$t = a - rp - s\tau + l, \qquad t = -a + (r+1)p + (s+1)\tau + l,$$

where $r,s \in \mathbb{Z}_{\geqslant 0}$, $l \in \mathbb{Z}$.

Hence the poles of the m-variable phase function $\Omega(t,z,\tau,p,\eta)$ are of first order and lie at the hyperplanes defined by equations

$$t_i - z_k = -a_k - rp - s\tau + l, \qquad t_i - z_k = a_k + (r+1)p + (s+1)\tau + l,$$

where $i = 1,\dots,m$, $k = 1,\dots,n$ and $r,s \in \mathbb{Z}_{\geqslant 0}$, $l \in \mathbb{Z}$, and

$$t_i = t_j + 2\eta - rp - s\tau + l, \qquad t_i = t_j - 2\eta + (r+1)p + (s+1)\tau + l,$$

where $1 \leqslant i < j \leqslant m$ and $r,s \in \mathbb{Z}_{\geqslant 0}$, $l \in \mathbb{Z}$.

The zeros of the m-variable phase function $\Omega(t,z,\tau,p,\eta)$ lie at the hyperplanes defined by equations

$$t_i - z_k = a_k - rp - s\tau + l, \qquad t_i - z_k = -a_k + (r+1)p + (s+1)\tau + l,$$

where $i = 1,\dots,m$, $k = 1,\dots,n$ and $r,s \in \mathbb{Z}_{\geqslant 0}$, $l \in \mathbb{Z}$, and

$$t_i = t_j - 2\eta - rp - s\tau + l, \qquad t_i = t_j + 2\eta + (r+1)p + (s+1)\tau + l,$$

where $1 \leqslant i < j \leqslant m$ and $r,s \in \mathbb{Z}_{\geqslant 0}$, $l \in \mathbb{Z}$.

The poles of the function $\omega_{l_1,\dots,l_n}(t,\lambda,z,\tau,\eta)$ are of first order and lie at the hyperplanes $t_i - z_k = a_k + s\tau + l$ where $i = 1,\dots,m$, $k = 1,\dots,n$, $s,l \in \mathbb{Z}$, and at the hyperplanes $t_i = t_j - 2\eta + s\tau + l$ where $1 \leqslant i < j \leqslant m$ and $s,l \in \mathbb{Z}$.

Similarly, the poles of the function $\omega_{m_1,\dots,m_n}(t,\mu,z,p,\eta)$ are of first order and lie at the hyperplanes $t_i - z_k = a_k + sp + l$ where $i = 1,\dots,m$, $k = 1,\dots,n$, $s,l \in \mathbb{Z}$, and at the hyperplanes $t_i = t_j - 2\eta + sp + l$ where $1 \leqslant i < j \leqslant m$ and $s,l \in \mathbb{Z}$.

Knowing the poles and zeros of the factors in (17), we get the Proposition. □

6.2. **Topology of poles.** The function defined by (17) depends on t, $z_1,\dots,z_n$, λ, μ, τ, p, η, $a_1,\dots,a_n$. Later on we often make the following assumptions on τ, p, η, $a_1,\dots,a_n$, $z_1,\dots,z_n$.

$$\operatorname{Im}\tau > 0\,, \qquad \operatorname{Im} p > 0\,, \qquad \operatorname{Im}\eta < 0\,. \tag{20}$$

$$\text{The numbers } \tau \text{ and } p \text{ are linearly independent over } \mathbb{Z}. \tag{21}$$

$$\{2\eta,\, 4\eta,\dots,\, 2m\eta\} \cap \{\mathbb{Z} + \tau\mathbb{Z} + p\mathbb{Z}\} \,=\, \varnothing\,. \tag{22}$$

$$2a_k + 2s\eta \notin \mathbb{Z} + \tau\mathbb{Z} + p\mathbb{Z}\,, \qquad k = 1,\dots,n\,, \qquad s = 1-m,\dots,m-1\,. \tag{23}$$

$$z_l \pm a_l - z_k \pm a_k + 2s\eta \notin \mathbb{Z} + \tau\mathbb{Z} + p\mathbb{Z}\,, \qquad k,l = 1,\dots,n\,, \quad l \neq k\,, \quad s = 1-m,\dots,m-1\,, \tag{24}$$

for arbitrary combination of signs.

A set of hyperplanes in an affine space is called a configuration of hyperplanes. An edge of a configuration is a nonempty intersection of some hyperplanes of the configuration.

Consider the configuration $\mathcal{B} = \mathcal{B}(\tau,p,\eta,a_1,\dots,a_n,z_1,\dots,z_n)$ of hyperplanes in $\mathbb{C}^m$ defined by equations (18) and (19).

Fix a natural number N and consider the image of the configuration $\mathcal{B}$ in $\mathbb{C}^m/N\mathbb{Z}^m$ under the natural projection $\mathbb{C}^m \to \mathbb{C}^m/N\mathbb{Z}^m$. We shall call the image a configuration of hyperplanes in $\mathbb{C}^m/N\mathbb{Z}^m$ and

denote the image by $\mathcal{C}_N = \mathcal{C}_N(\tau, p, \eta, a_1, \dots, a_n, z_1, \dots, z_n)$. We always assume that $\mathrm{Im}\,\tau > 0$, $\mathrm{Im}\,p > 0$, and therefore $\mathcal{C}_N$ is a locally finite collection of hyperplanes in $\mathbb{C}^m/N\mathbb{Z}^m$.

Let $F(t, \lambda, \mu, z, a, \tau, p)$ be the function defined by (17). The function is 1-periodic in all variables t_i, and therefore it defines a function on $\mathbb{C}^m/N\mathbb{Z}^m$ which we denote also by $F(t, \lambda, \mu, z, a, \tau, p)$. The function $F(t, \lambda, \mu, z, a, \tau, p)$ is holomorphic on the complement to the union of hyperplanes of $\mathcal{C}_N$ in $\mathbb{C}^m/N\mathbb{Z}^m$. Moreover, for any $\beta \in \mathbb{Z}^m + \tau\mathbb{Z}^m + p\mathbb{Z}^m$, the function $F(t+\beta, \lambda, \mu, z, a, \tau, p)$ is holomorphic on the complement to the union of hyperplanes of $\mathcal{C}_N$ in $\mathbb{C}^m/N\mathbb{Z}^m$.

Proposition 26. *The number of pairwise distinct edges of any dimension and the dimensions of all edges of the configuration $\mathcal{C}_N$ do not depend on the parameters $\tau, p, \eta, a_1, \dots, a_n, z_1, \dots, z_n$ provided that assumptions (20) – (24) hold.*

Proof: The initial configuration induces a configuration of hyperplanes in any edge of the initial configuration. The number of pairwise distinct edges of any dimension and the dimensions of all edges of the initial configuration remain the same if for each induced configuration the number of its pairwise distinct hyperplanes does not change. This is obviously true if assumptions (20) – (24) hold. □

Corollary 27. *The topology of the complement of the configuration of hyperplanes $\mathcal{C}_N$ in $\mathbb{C}^m/N\mathbb{Z}^m$ remains the same for all τ, p, η, $z_1, \dots, z_n$, $a_1, \dots, a_n$ satisfying conditions (20) – (24).*

The Corollary follows from standard reasons in topological singularity theory, cf. [R].

6.3. **Hypergeometric integrals.** Let $\tau, p, \eta, a_1, \dots, a_n$ be complex numbers. Assume that (20) holds and

$$\mathrm{Im}\, a_k > 0\,, \qquad k = 1, \dots, n\,. \tag{25}$$

Let $t = (t_1, \dots, t_m)$, $z = (z_1, \dots, z_n)$, $a = (a_1, \dots, a_n)$. Let $\Omega(t, z, a, \tau, p, \eta)$ be the m-variable phase function introduced in (13). Let $\omega_L(t, \lambda, z, a, \tau, \eta)$ and $\omega_M(t, \mu, z, a, p, \eta)$ be the functions introduced in (3), here $L = (l_1, \dots, l_n)$, $l_1 + \ldots + l_n = m$, and $M = (m_1, \dots, m_n)$, $m_1 + \ldots + m_n = m$. Let $\varphi(t, \lambda, \mu, z, a, \tau, p, \eta)$ be a holomorphic function for $t \in \mathbb{C}^m$, $z, a \in \mathbb{C}^n$, and τ, p lying in the upper half plane. Assume that φ is N-periodic in t, $\varphi(t+\beta, \lambda, \mu, z, a, \tau, p, \eta) = \varphi(t, \lambda, \mu, z, a, \tau, p, \eta)$ for all $\beta \in N\mathbb{Z}^m$.

Consider the integral

$$I^{\varphi}_{LM}(\lambda, \mu, z, a, \tau, p, \eta) \, = \tag{26}$$

$$\int_{[0,N]^m} \Omega(t, z, a, \tau, p, \eta)\, \omega_L(t, \lambda, z, a, \tau, \eta)\, \omega_M(t, \mu, z, a, p, \eta)\, \varphi(t, \lambda, \mu, z, a, \tau, p, \eta)\, dt$$

where $dt = dt_1 \wedge \ldots \wedge dt_n$. The integral will be called *a hypergeometric integral*.

The integrand, $F(t, \lambda, \mu, z, a, \tau, p, \eta)$, in (26) is N-periodic, hence the integral can be considered as an integral over the image in $\mathbb{C}^m/N\mathbb{Z}^m$ of the subspace $\mathbb{R}^m$ under the natural projection.

For any $\beta \in \mathbb{C}^m$, denote by $I^{\varphi}_{LM}(\lambda, \mu, z, a, \tau, p, \eta)_\beta$ the integral

$$\int_{[0,N]^m} F(t+\beta, \lambda, \mu, z, a, \tau, p)\, dt\,. \tag{27}$$

Fix $\beta \in \tau\mathbb{Z}^m + p\mathbb{Z}^m$ and $z \in \mathbb{C}^n$. Assume that for all $i = 1, \dots, n$,

$$\mathrm{Im}\, a_i \gg \mathrm{Im}\,\tau\,, \quad \mathrm{Im}\, a_i \gg \mathrm{Im}\, p\,, \quad \text{and} \quad -\mathrm{Im}\,\eta \gg \mathrm{Im}\,\tau\,, \quad -\mathrm{Im}\,\eta \gg \mathrm{Im}\, p\,. \tag{28}$$

Then it is easy to see that the poles of the integrand of (27) lie far from the integration cycle $[0, N]^m$. Hence the integral is well defined and holomorphically depends on the parameters $\lambda, \mu, z, a, \tau, p, \eta$. We shall call this range of parameters *the starting range for given β and z.*

Proposition 28. *For fixed $\beta, \gamma \in \tau\mathbb{Z}^m + p\mathbb{Z}^m$ and $z \in \mathbb{C}^n$, the integrals $I^{\varphi}_{LM}(\lambda, \mu, z, a, \tau, p, \eta)_\beta$ and $I^{\varphi}_{LM}(\lambda, \mu, z, a, \tau, p, \eta)_\gamma$ are equal if condition (28) hold.*

Proof: Change variables, $t \to t' - \beta + \gamma$, in the first integral. Then the integrand of the first integral becomes equal to the integrand of the second while the integration cycle becomes equal to $[0, N]^m + \beta - \gamma$. If condition (28) hold, then the integrand has no poles around the tori $[0, N]^m + \beta - \gamma$ and $[0, N]^m$. Since the integrand is a closed differential form, the integrals over $[0, N]^m + \beta - \gamma$ and $[0, N]^m$ are equal. □

So far we defined the function $I^{\varphi}_{LM}(\lambda, \mu, z, a, \tau, p, \eta)_\beta$ in the starting range of parameters with respect to given β and z as the integral of $F(t+\beta, \lambda, \mu, z, a, \tau, p)\, dt$ over the torus $[0, N]^m \subset \mathbb{C}^m/N\mathbb{Z}^m$. Moreover, this function does not depend on β. In order to define the function $I^{\varphi}_{LM}(\lambda, \mu, z, a, \tau, p, \eta)_\beta$ for all values of parameters satisfying conditions (20) – (24) we use analytic continuation. The result of the analytic

continuation can be represented as an integral of the integrand over a suitably deformed torus, which we denote by T^m_N. Namely, the poles of the integrand are located at the hyperplanes of the configuration $\mathcal{C}_N(\tau, p, \eta, a_1, \dots, a_n, z_1, \dots, z_n)$. We deform the parameters $\tau,\, p,\, \eta,\, a_1, \dots, a_n,\, z_1, \dots, z_n$ preserving conditions (20) – (24). Then by Corollary 27, the topology of the complement of $\mathcal{C}_N$ in $\mathbb{C}^m/N\mathbb{Z}^m$ does not change. We deform the integration torus accordingly so that the torus does not intersect the hyperplanes of the configuration at every moment of the deformation. Then the analytic continuation of the function $I^{\varphi}_{LM}(\lambda, \mu, z, a, \tau, p, \eta)_\beta$ is given by the integral

$$\int_{T^m_N} F(t+\beta, \lambda, \mu, z, a, \tau, p)\, dt\,. \tag{29}$$

Theorem 29. *The integral (27) can be analytically continued as a holomorphic univalued function to the domain of the parameters $\lambda, \mu, z, a, \tau, p, \eta$ satisfying conditions (20) – (24).*

The proof of the Theorem is the same as the proof of Theorem 5.7 in [TV1].

We have the following important corollary.

Corollary 30. *The function $I^{\varphi}_{LM}(\lambda, \mu, z, a, \tau, p, \eta)_\beta$ defined by (29) does not depend on β.*

6.4. **Solutions to qKZB.** Fix complex numbers $\tau,\, p,\, \eta,\, \Lambda_1, \dots, \Lambda_n$ and set $a_i = \eta\Lambda_i$. Assume that the parameters $\tau,\, p,\, \eta,\, a_1, \dots, a_n$ satisfy conditions (20) – (23) and $\Lambda_1 + \ldots + \Lambda_n = m$ for some positive integer m.

Let $\Omega(t, z, a, \tau, p, \eta)$ be the m-variable phase function introduced in (13). Let $\omega_L(t, \lambda, z, a, \tau, \eta)$ and $\omega_M(t, \mu, z, a, p, \eta)$ be the functions introduced in (3), here $L = (l_1, \dots, l_n)$, $|L| = l_1 + \ldots + l_n = m$, and $M = (m_1, \dots, m_n)$, $|M| = m$.

Fix a natural number N. Let ξ be an entire function of one variable which is $4\eta N$-periodic, $\xi(\lambda + 4\eta N) = \xi(\lambda)$.

Let $V_\Lambda = \oplus_{j=0}^{\infty} \mathbb{C} e_j$ and $V = V_{\Lambda_1} \otimes \ldots \otimes V_{\Lambda_n}$. For any $L = (l_1, \dots, l_n)$, $|L| = m$, set $e_L = e_{l_1} \otimes \ldots \otimes e_{l_n} \in V$.

Introduce a $V[0] \otimes V[0]$-valued function u^ξ by

$$u^\xi(z_1, \dots, z_n, \lambda, \mu, \tau, p) \,=\, \sum_{L, M,\, |L|=|M|=m} u^\xi_{LM}(z_1, \dots, z_n, \lambda, \mu, \tau, p)\, e_L \otimes e_M \tag{30}$$

where

$$u^\xi_{LM}(z_1, \dots, z_n, \lambda, \mu, \tau, p) \,=\, e^{-\pi i \frac{\mu\lambda}{2\eta}} \,\times \tag{31}$$

$$\int_{T^m_N} \xi(p\lambda + \tau\mu - \sum_{l=1}^{n} 2a_l z_l + 4\eta \sum_{j=1}^{m} t_j)\, \Omega(t, z, a, \tau, p, \eta)\, \omega_L(t, \lambda, z, a, \tau, \eta)\, \omega_M(t, \mu, z, a, p, \eta)\, dt\,.$$

Here we assume that $z_1, \dots, z_n$ satisfy condition (24) and we define the integral by analytic continuation described in Section 6.3.

For any $j = 1, \dots, n$, introduce an End $(V[0])$-valued function $D_j(\lambda)$ by the formula

$$D_j(\lambda) : e_L \,\mapsto\, \frac{\alpha(\lambda \,-\, 2\eta \sum_{l=1}^{j} h^{(l)})}{\alpha(\lambda \,-\, 2\eta \sum_{l=1}^{j-1} h^{(l)})}\, e^{\pi i \eta \Lambda_j (\sum_{l=1}^{j-1} \Lambda_l - \sum_{l=j+1}^{n} \Lambda_l)}\, e_L\,, \tag{32}$$

cf. (6), and an End $(V[0])$-valued function $D(\lambda, z, p)$ by

$$D(\lambda, z, p) \,=\, \prod_{j=1}^{n} D_j(\mu)^{z_j/p}\,, \tag{33}$$

cf. (14).

Theorem 31. *Under the above conditions, for any $j = 1, \dots, n$, we have*

$$u(\dots, z_j + p, \dots) = K_j(z_1, \dots, z_n, \lambda, \tau, p) \otimes D_j^{-1}(\mu)\; u(\dots, z_j, \dots), \tag{34}$$

$$u(\dots, z_j + \tau, \dots) = D_j^{-1}(\lambda) \otimes K_j(z_1, \dots, z_n, \mu, p, \tau)\; u(\dots, z_j, \dots), \tag{35}$$

and, if in addition the function ξ is $2a_l$-periodic for all $l = 1, \dots, n$, then

$$u(\dots, z_j + 1, \dots) = u(\dots, z_j, \dots). \tag{36}$$

Here $K_j(z_1, \dots, z_n, \lambda, \tau, p)$ is the qKZB operator defined by (1), i.e. the operator of the qKZB equations with step p and defined in terms of elliptic R-matrices with modulus τ.

The following Corollary is equivalent to the first two statements of the Theorem. The third statement is trivial.

Corollary 32. *Let $f : V \to \mathbb{C}$ be a linear function. Consider the functions*

$$\Psi_f^\xi(z,\lambda,\mu,\tau,p) \,=\, (1\otimes f)\,(1\otimes D(\mu,z,p))\,u(z,\lambda,\mu,\tau,p)\,, \tag{37}$$

and

$$\Phi_f^\xi(z,\lambda,\mu,\tau,p) \,=\, (f\otimes 1)\,(D(\lambda,z,\tau)\otimes 1)\,u(z,\lambda,\mu,\tau,p)\,. \tag{38}$$

Then for a fixed μ, the function Ψ^ξ is a solution of the qKZB equations with modulus τ and step p,

$$\begin{aligned}\Psi_f^\xi(z_1,\dots,z_j+p,\dots,z_n,\lambda,\mu,\tau,p) \,=&\\ =\, K_j(z_1,\dots,z_n,\lambda,\tau,p)\,\Psi_f^\xi(z_1,\dots,z_n,\lambda,\mu,\tau,p)\,,& \qquad j=1,\dots,n\,,\end{aligned} \tag{39}$$

and for a fixed λ, the function Φ^ξ is a solution of the qKZB equations with modulus p and step τ,

$$\begin{aligned}\Phi_f^\xi(z_1,\dots,z_j+\tau,\dots,z_n,\lambda,\mu,\tau,p) \,=&\\ = K_j(z_1,\dots,z_n,\mu,p,\tau)\,\Psi_f^\xi(z_1,\dots,z_n,\lambda,\mu,\tau,p)\,,& \qquad j=1,\dots,n\,.\end{aligned} \tag{40}$$

Proof: Fix $M=(m_1,\dots,m_n)$, $|M|=m$. By Theorem 23, the function Ψ defined by (15) is a formal solution to the qKZB equations (39). This means that for any j, the difference

$$\Psi^\xi(z_1,\dots,z_j+p,\dots,z_n,\lambda,\mu,\tau,p)-K_j(z_1,\dots,z_n,\lambda,\tau,p)\,\Psi^\xi(z_1,\dots,z_n,\lambda,\mu,\tau,p)$$

is an element of the space $DE_a(z;\xi;M)\otimes V[0]$. By Corollary 30, the integral in (31), defined by the analytic continuation described in Section 6.3, is equal to zero on $DE_a(z;\xi;M)\otimes V[0]$. Hence, if $f:V\to\mathbb{C}$ is a linear function, then, for a fixed μ, the function Ψ_f^ξ, defined by (37), is a solution to the qKZB equations (39). This proves the first statement of Corollary 32. The proof of the second statement is similar. The Corollary implies the Theorem. □

The solutions of the qKZB equations constructed in Corollary 32 depend on an entire function ξ which is $4\eta N$-periodic. An important example of such a function is the function $\xi\equiv 1$. In the next section we consider another important example of entire functions.

6.5. **Theta function properties of solutions.** Recall that a scalar theta function of level N is a function f such that $f(\lambda+1)=f(\lambda)$, $f(\lambda+\tau)=e^{-\pi i N(2\lambda+\tau)}f(\lambda)$.

Consider the objects described in Section 6.4, in particular, let $V=V_{\Lambda_1}\otimes\ldots\otimes V_{\Lambda_n}$ as in Section 6.4. We say that a $V[0]$-valued function $\Psi(\lambda)$ is a vector-valued theta function of level N, if

$$\Psi(\lambda+1) \,=\, \Psi(\lambda), \tag{41}$$

$$\Psi(\lambda+\tau) \,=\, e^{-\pi i(2N\lambda+N\tau+\sum_{j=1}^n h^{(j)}(z_j-a_j-2a_{j+1}-\ldots-2a_n))}\,\Psi(\lambda). \tag{42}$$

Clearly, the space of holomorphic vector-valued theta functions of level N is finite-dimensional.

Assume that the numbers p and η are such that $-p/4\eta=N$ where N is a natural number. Let $f(\lambda)$ be a scalar theta function of level N. Set $\xi(\lambda)=f(\lambda/p)$. Then ξ is an entire $4\eta N$-periodic function.

Consider the solution $\Psi_f^\xi(z,\lambda,\mu,\tau,p,\eta)$ to the qKZB equations constructed in Corollary 32.

Theorem 33. *Under the above assumptions, if the parameter μ has the form $\mu=2\eta(m+2s)$, $s\in\mathbb{Z}$, then the solution $\Psi_f^\xi(z,\lambda,\mu,\tau,p,\eta)$ is a vector-valued theta function of level $m+N$ as a function of variable λ.*

Proof: We have $\Psi_f^\xi(\dots,\lambda+1,\dots)=e^{-\pi i\mu/2\eta}(-1)^m\Psi_f^\xi(\dots,\lambda,\dots)$ where the first factor comes from the factor $e^{-\pi i\frac{\mu\lambda}{2\eta}}$ in (31) and the second factor comes from $\omega_L(\dots,\lambda,\dots)$ in (31). Hence, according to our assumptions, $\Psi_f^\xi(\dots,\lambda,\dots)$ satisfies (41).

In order to check property (42) we shall check the property for each coordinate function u_{LM}^ξ defined by (31).

We have $u_{LM}^\xi(\dots,\lambda+\tau,\dots)=e^{-\pi iA}e^{-\pi iB}e^{-\pi iC}u_{LM}^\xi(\dots,\lambda,\dots)$ where the first, second and third factors correspond to transformations of $e^{-\pi i\mu/2\eta}$, ξ and ω_L in (31), respectively. Here $A=\mu\tau/2\eta$, $B=N(2\lambda+2\tau\mu/p-4\eta/p\sum\Lambda_k z_k+8\eta/p\sum t_j+\tau)$,

$$C \,=\, m\tau \,+\, \sum_{k=1}^n\ \sum_{j=l_1+\ldots+l_{k-1}+1}^{l_1+\ldots+l_k} 2\big(\lambda+t_j-z_k-a_k+2l_k\eta-2\eta\sum_{q=1}^{k-1}(\Lambda_q-2l_q)\big)\,.$$

Using our assumptions, after simple calculations we get

$$A+B+C=2(N+m)\lambda+(N+m)\tau+\sum_{j=1}^{n} h^{(j)}(z_j-a_j-2a_{j+1}-\ldots-2a_n)$$

where $h^{(j)}=\Lambda_j-2l_j$. The Theorem is proved. $\square$

7. Monodromy of solutions of qKZB equations

7.1. Monodromy with respect to permutations of variables.

Theorem 34. *Let $\Psi(z_1,\dots,z_n,\lambda)$ be a solution of the qKZB equations with values in $(V_{\Lambda_1}\otimes\ldots\otimes V_{\Lambda_n})[0]$ step p and modulus τ. Then for any $j=1,\dots,n-1$, the function*

$$\Psi_j(z_1,\dots,z_n,\lambda)\;=\;P^{(j,j+1)}\,R^{(j,j+1)}_{\Lambda_j,\Lambda_{j+1}}(z_{j+1}-z_j,\lambda-2\eta\sum_{l=1}^{j-1}h^{(l)},\tau)\,\Psi(z_1,\dots,z_{j+1},z_j,\dots,z_n,\lambda)\qquad(43)$$

is a solution of the qKZB equations with values in $(V_{\Lambda_1}\otimes\ldots V_{\Lambda_{j+1}}\otimes V_{\Lambda_j}\ldots\otimes V_{\Lambda_n})[0]$, step p and modulus τ. Here $P^{(j,j+1)}$ is the permutation of the j-th and $j{+}1$-th factors, and $R_{\Lambda_j,\Lambda_{j+1}}(z,\lambda,\tau)\in\mathrm{End}\,(V_{\Lambda_j}\otimes V_{\Lambda_{j+1}})$ is the elliptic R-matrix with modulus τ.

Proof: The proof of the Theorem is straightforward. Let us check, for instance, that Ψ_j satisfies the first qKZB equation assuming that $j>1$. We have

$$\begin{aligned}
&\Psi_j(z_1+p,\ldots)\;=\;P^{(j,j+1)}\,R^{(j,j+1)}_{\Lambda_j,\Lambda_{j+1}}(z_{j+1}-z_j,\lambda-2\eta\sum_{l=1}^{j-1}h^{(l)},\tau)\\
&\times K_1^{\Lambda_1,\dots,\Lambda_n}(z_1,\dots,z_{j+1},z_j,\dots,z_n,\lambda)\,\Psi(z_1,\dots,z_{j+1},z_j,\dots)\;=\\
&=\;P^{(j,j+1)}\,R^{(j,j+1)}_{\Lambda_j,\Lambda_{j+1}}(z_{j+1}-z_j,\lambda-2\eta\sum_{l=1}^{j-1}h^{(l)},\tau)\,\Gamma_1\,R^{(1,n)}_{\Lambda_1,\Lambda_n}(z_1-z_n,\lambda-2\eta\sum_{l=2}^{n-1}h^{(l)},\tau)\,\ldots\\
&\times R^{(1,j+1)}_{\Lambda_1,\Lambda_{j+1}}(z_1-z_j,\lambda-2\eta\sum_{l=2}^{j}h^{(l)},\tau)\,R^{(1,j)}_{\Lambda_1,\Lambda_j}(z_1-z_{j+1},\lambda-2\eta\sum_{l=2}^{j-1}h^{(l)},\tau)\,\ldots\\
&\times R^{(1,2)}_{\Lambda_1,\Lambda_2}(z_1-z_2,\lambda,\tau)\,\Psi(z_1,\dots,z_{j+1},z_j,\dots)\;=\\
&=\;P^{(j,j+1)}\,\Gamma_1\,R^{(1,n)}_{\Lambda_1,\Lambda_n}(z_1-z_n,\lambda-2\eta\sum_{l=2}^{n-1}h^{(l)},\tau)\,\ldots\,R^{(j,j+1)}_{\Lambda_j,\Lambda_{j+1}}(z_{j+1}-z_j,\lambda-2\eta\sum_{l=2}^{j-1}h^{(l)},\tau)\\
&\times R^{(1,j+1)}_{\Lambda_1,\Lambda_{j+1}}(z_1-z_j,\lambda-2\eta\sum_{l=2}^{j}h^{(l)},\tau)\,R^{(1,j)}_{\Lambda_1,\Lambda_j}(z_1-z_{j+1},\lambda-2\eta\sum_{l=2}^{j-1}h^{(l)},\tau)\,\ldots\\
&\times R^{(1,2)}_{\Lambda_1,\Lambda_2}(z_1-z_2,\lambda,\tau)\,\Psi(z_1,\dots,z_{j+1},z_j,\dots)\;=\\
&=\;P^{(j,j+1)}\,\Gamma_1\,R^{(1,n)}_{\Lambda_1,\Lambda_n}(z_1-z_n,\lambda-2\eta\sum_{l=2}^{n-1}h^{(l)},\tau)\,\ldots\,R^{(1,j)}_{\Lambda_1,\Lambda_j}(z_1-z_{j+1},\lambda-2\eta\sum_{l=2,\,l\neq j}^{j+1}h^{(l)},\tau)\,\ldots\\
&\times R^{(1,j+1)}_{\Lambda_1,\Lambda_{j+1}}(z_1-z_j,\lambda-2\eta\sum_{l=2}^{j-2}h^{(l)},\tau)\,R^{(j,j+1)}_{\Lambda_j,\Lambda_{j+1}}(z_{j+1}-z_j,\lambda-2\eta\sum_{l=1}^{j-1}h^{(l)},\tau)\\
&\times R^{(1,2)}_{\Lambda_1,\Lambda_2}(z_1-z_2,\lambda,\tau)\,\Psi(z_1,\dots,z_{j+1},z_j,\dots)\;=\\
&=\;K_1^{\Lambda_1,\dots,\Lambda_{j+1},\Lambda_j,\dots,\Lambda_n}(z_1,\dots,z_j,z_{j+1},\dots,z_n,\lambda)\,\Psi_j(z_1,\dots)\,.
\end{aligned}$$

The other cases are checked similarly. $\square$

Remark. The transformation

$$T_j:\Psi(z_1,\dots,z_n,\lambda)\to P^{(j,j+1)}\,R^{(j,j+1)}_{\Lambda_j,\Lambda_{j+1}}(z_{j+1}-z_j,\lambda-2\eta\sum_{l=1}^{j-1}h^{(l)},\tau)\,\Psi(z_1,\dots,z_{j+1},z_j,\dots,z_n,\lambda)$$

preserves the theta function properties (41), (42).

Proposition 35. *Assume that a $(V_{\Lambda_1}\otimes\ldots\otimes V_{\Lambda_n})[0]$-valued function $\Psi(z_1,\dots,z_n,\lambda)$ satisfies the theta function properties (41), (42). Then for any $j=1,\dots,n$, the $(V_{\Lambda_1}\otimes\ldots V_{\Lambda_{j+1}}\otimes V_{\Lambda_j}\ldots\otimes V_{\Lambda_n})[0]$-valued function $T_j\Psi$ also satisfies the theta function properties (41), (42).*

The Proposition easily follows from Proposition 13.

Let $u^{\xi,\Lambda_1,\dots,\Lambda_n}(z_1,\dots,z_n,\lambda,\mu,\tau,p)$ be the function constructed for the tensor product $V = V_{\Lambda_1}\otimes \dots\otimes V_{\Lambda_n}$ in (30). Let $D_{\Lambda_1,\dots,\Lambda_n}(\mu,z,p)$ be the End$\,(V[0])$-valued function defined in (33). According to Theorem 31, the $V[0]\otimes V[0]$-valued function

$$\Psi^{\Lambda_1,\dots,\Lambda_n}(\mu,z) \,=\, (1\otimes D_{\Lambda_1,\dots,\Lambda_n}(\mu,z,p))\, u^{\xi,\Lambda_1,\dots,\Lambda_n}(z_1,\dots,z_n,\lambda,\mu,\tau,p) \tag{44}$$

is a solution of the qKZB equations with respect to the first factor,

$$\Psi^{\Lambda_1,\dots,\Lambda_n}(\dots,z_k+p,\dots) \,=\, (K_k(z_1,\dots,z_n,\lambda,\tau,p)\otimes 1)\ \Psi^{\Lambda_1,\dots,\Lambda_n}(\dots,z_k,\dots)\,,\qquad k=1,\dots,n.$$

For any $j=1,\dots,n$, denote by V_j the tensor product $V_{\Lambda_1}\otimes\dots V_{\Lambda_{j+1}}\otimes V_{\Lambda_j}\dots\otimes V_{\Lambda_n}$. According to Theorem 34, for any j, the $V[0]\otimes V_j[0]$-valued function

$$\begin{aligned}\Psi_j(z,\lambda,\mu,\tau,p) \,&=\, (P^{(j,j+1)}R^{(j,j+1)}_{\Lambda_{j+1},\Lambda_j}(z_{j+1}-z_j,\lambda-2\eta\sum_{l=1}^{j-1}h^{(l)},\tau)\otimes\\ &D_{\dots,\Lambda_{j+1},\Lambda_j,\dots}(\mu,\dots,z_{j+1},z_j,\dots,p))\,u^{\xi,\dots,\Lambda_{j+1},\Lambda_j,\dots}(\dots,z_{j+1},z_j,\dots)\end{aligned}$$

is a solution with respect to the first factor of the same qKZB equations.

The next Theorem describes a relation between the two solutions and can be considered as a description of the monodromy of the hypergeometric solutions constructed in Section 6.4 with respect to permutation of variables.

Introduce a new R-matrix $\tilde R_{A,B}(z,\mu,p)\in \mathrm{End}\,(V_A\otimes V_B)$ by

$$\tilde R_{A,B}(z,\mu,p) \,=\, e^{2\pi i ABz/p}\Big(\frac{\alpha(\lambda)}{\alpha(\lambda-2\eta h^{(2)})}\Big)^{z/p} R_{A,B}(z,\mu,p)\Big(\frac{\alpha(\lambda-2\eta(h^{(1)}+h^{(2)})}{\alpha(\lambda-2\eta h^{(1)})}\Big)^{z/p}. \tag{45}$$

Theorem 36.

$$\Psi_j(z,\lambda,\mu,\tau,p) \,=\, (1\otimes P^{(j,j+1)}\tilde R^{(j,j+1)}_{\Lambda_j,\Lambda_{j+1}}(z_j-z_{j+1},\mu-2\eta\sum_{l=1}^{j-1}h^{(l)},p))\Psi^{\Lambda_1,\dots,\Lambda_n}(z,\lambda,\mu,\tau,p)\,. \tag{46}$$

Remark. According to Proposition 12, the matrix $\tilde R_{A,B}(z,\mu,p)$ is p-periodic,

$$\tilde R_{A,B}(z+p,\mu,p) \,=\, \tilde R_{A,B}(z,\mu,p)\,.$$

Hence, formula (46) expresses solution Ψ_j as a linear combination of solutions $\Psi^{\Lambda_1,\dots,\Lambda_n}$ with p-periodic coefficients.

Proof: First let us prove that

$$\begin{aligned}\Psi_j(z,\lambda,\mu,\tau,p) \,&=\, \big(1\otimes D_{\dots,\Lambda_{j+1},\Lambda_j,\dots}(\mu,\dots,z_{j+1},z_j,\dots,p)\\ &\times P^{(j,j+1)}\,R^{(j,j+1)}_{\Lambda_j,\Lambda_{j+1}}(z_j-z_{j+1},\mu-2\eta\sum_{l=1}^{j-1}h^{(l)},p)\\ &\times D^{-1}_{\dots,\Lambda_j,\Lambda_{j+1},\dots}(\mu,\dots,z_j,z_{j+1},\dots,p)\big)\,\Psi^{\Lambda_1,\dots,\Lambda_n}(z,\lambda,\mu,\tau,p)\,.\end{aligned} \tag{47}$$

In fact, we have

$$\begin{aligned}\Psi_j(z,\lambda,\mu,\tau,p) \,&=\, (P^{(j,j+1)}R^{(j,j+1)}_{\Lambda_{j+1},\Lambda_j}(z_{j+1}-z_j,\lambda-2\eta\sum_{l=1}^{j-1}h^{(l)},\tau)\otimes\\ D_{\dots,\Lambda_{j+1},\Lambda_j,\dots}&(\mu,\dots,z_{j+1},z_j,\dots))\sum_{L,M,\,|L|=|M|=m}u_{LM}(\dots,z_{j+1},z_j,\dots)\,e_L\otimes e_M\end{aligned} \tag{48}$$

where $L=(l_1,\dots,l_{j+1},l_j,\dots,l_n)$, $M=(m_1,\dots,m_{j+1},m_j,\dots,m_n)$, and

$$\begin{aligned}u_{LM}(\dots,z_{j+1},z_j,\dots) \,&=\, e^{-\pi i\frac{\mu\lambda}{2\eta}}\int_{T^m_N}\xi(p\lambda+\tau\mu-\sum_{l=1}^{n}2a_lz_l+4\eta\sum_{j=1}^{m}t_j)\,\Omega(t,z,a,\tau,p,\eta)\\ &\times\,\omega_L^{\dots,\Lambda_{j+1},\Lambda_j,\dots}(t,\lambda,\dots,z_{j+1},z_j,\dots,\tau)\,\omega_M^{\dots,\Lambda_{j+1},\Lambda_j,\dots}(t,\mu,\dots,z_{j+1},z_j,\dots,p)\,dt\,.\end{aligned}$$

According to (4), we have

$$\begin{aligned}P^{(j,j+1)}\sum_L\omega_L^{\dots,\Lambda_{j+1},\Lambda_j,\dots}(t,\lambda,\dots,z_{j+1},z_j,\dots,\tau)\,R^{(j,j+1)}_{\Lambda_{j+1},\Lambda_j}(z_{j+1}-z_j,\lambda-2\eta\sum_{l=1}^{j-1}h^{(l)},\tau)\,e_L\\ =\sum_S\omega_S^{\dots,\Lambda_j,\Lambda_{j+1},\dots}(t,\lambda,\dots,z_j,z_{j+1},\dots,\tau)\,e_S\end{aligned} \tag{49}$$

where $e_S \in V$ and

$$\sum_M \omega_M^{\dots,\Lambda_{j+1},\Lambda_j,\dots}(t,\mu,\dots,z_{j+1},z_j,\dots,p)\,e_M \;= \tag{50}$$

$$=\; P^{(j,j+1)} \sum_Q \omega_Q^{\dots,\Lambda_j,\Lambda_{j+1},\dots}(t,\mu,\dots,z_j,z_{j+1},\dots,p)\,R^{(j,j+1)}_{\Lambda_j,\Lambda_{j+1}}(z_j-z_{j+1},\mu-2\eta\sum_{l=1}^{j-1}h^{(l)},p)\,e_Q$$

where $e_Q \in V$. Formulae (49), (50) and (48) prove (47).

Now the Theorem follows from the following Lemma.

Lemma 37.

$$D_{\dots,\Lambda_{j+1},\Lambda_j,\dots}(\mu,\dots,z_{j+1},z_j,\dots,p)P^{(j,j+1)}R^{(j,j+1)}_{\Lambda_j,\Lambda_{j+1}}(z_j-z_{j+1},\mu-2\eta\sum_{l=1}^{j-1}h^{(l)},p)$$

$$\times\, D^{-1}_{\dots,\Lambda_j,\Lambda_{j+1},\dots}(\mu,\dots,z_j,z_{j+1},\dots,p))\;=\;P^{(j,j+1)}\tilde R^{(j,j+1)}_{\Lambda_j,\Lambda_{j+1}}(z_j-z_{j+1},\mu-2\eta\sum_{l=1}^{j-1}h^{(l)},p)$$

The Lemma follows from (33) and (32). □

7.2. **Monodromy of solutions with respect to shifts of z_j by τ.** Let $u(z_1,\dots,z_n,\lambda,\mu,\tau,p)$ be the function constructed for the tensor product $V = V_{\Lambda_1}\otimes\dots\otimes V_{\Lambda_n}$ in (30). Let $D(\mu,z,p)$ be the $\mathrm{End}\,(V[0])$-valued function defined in (33). Set

$$\Psi(z,\lambda,\mu,\tau,p)\;=\;(1\otimes D(\mu,z,p))\,u(z_1,\dots,z_n,\lambda,\mu,\tau,p)\,.$$

According to Theorem 31, the $V[0]\otimes V[0]$-valued function Ψ is a solution of the qKZB equations with respect to the first factor,

$$\Psi(\dots,z_k+p,\dots)\;=\;(K_k(z,\lambda,\tau,p)\otimes 1)\;\Psi(\dots,z_k,\dots),\qquad k=1,\dots,n\,.$$

Let $B_j(z,\lambda,p)\in\mathrm{End}\,(V[0])$ be the linear operator introduced in (9). According to Theorem 16, for any $j=1,\dots,n$, the function

$$\Psi_j(z,\lambda,\mu,\tau,p)\;=\;(B_j(z,\lambda,p)\otimes 1)\,\Psi(z_1,\dots,z_j+\tau,\dots,z_n,\lambda,\mu,\tau,p)$$

is a new solution of the same equations.

The next Theorem describes a relation between the two solutions and can be considered as a description of the monodromy of the hypergeometric solutions constructed in Section 6.4 with respect to shifts of variables z_j by τ.

Theorem 38.

$$\begin{aligned}\Psi_j(z,\lambda,\mu,\tau,p)\;=\;&\big(1\,\otimes\,f_j(z,p)\\ &\times\,D(\dots,z_j+\tau,\dots,\mu,p)\,K_j(z,\mu,p,\tau)\,D^{-1}(\dots,z_j,\dots,\mu,p)\big)\,\Psi(z,\lambda,\mu,\tau,p)\end{aligned}\tag{51}$$

where $f_j(z,p)=e^{2\pi i\eta\sum_{l,\,l\neq j}(z_l-z_j)\Lambda_l\Lambda_j/p}$ and $K_j(z,\mu,p,\tau)$ is the j-th operator of the qKZB equations with step τ and modulus p.

Remark. According to Theorem 15, the operator

$$\tilde K_j(z,\mu,p,\tau)\;=\;f_j(z,p)D(\dots,z_j+\tau,\dots,\mu,p)\,K_j(z,\mu,p,\tau)\,D^{-1}(\dots,z_j,\dots,\mu,p)$$

is p-periodic, $\tilde K_j(\dots,z_k+p,\dots)=\tilde K_j(\dots,z_k,\dots)$. Hence, formula (51) expresses solution Ψ_j as a linear combination of solutions Ψ with p-periodic coefficients.

Proof:

$$\begin{aligned}&\Psi_j(z,\lambda,\mu,\tau,p)\;=\;(B_j(z,\lambda,p)\otimes 1)\,\Psi(\dots,z_j+\tau,\dots)\;=\\ &(f_j(z,p)\,D^{-1}(z,\lambda,\tau)D(\dots,z_j+\tau,\dots,\lambda,\tau)\otimes D(\dots,z_j+\tau,\dots,\mu,p))\,u(\dots,z_j+\tau,\dots)\;=\\ &(f_j(z,p)\,D^{-1}(z,\lambda,\tau)\,D(z,\lambda,\tau)\otimes D(\dots,z_j+\tau,\dots,\mu,p)\,K_j(z,\mu,p,\tau))\,u(\dots,z_j,\dots)\;=\\ &(1\otimes f_j(z,p)\,D(\dots,z_j+\tau,\dots,\mu,p)\,K_j(z,\mu,p,\tau)\,D^{-1}(z,\mu,p))\,\Psi(\dots,z_j,\dots)\,.\end{aligned}$$

□

Remark. Assume that $\Lambda_1,\dots,\Lambda_n$ are natural numbers, then the square of the transformation

$$T_j:\Psi(z_1,\dots,z_n,\lambda)\;\to\;B_j(z,\lambda,p)\,\Psi(z_1,\dots,z_j+\tau,\dots,z_n,\lambda)$$

preserves the theta function properties (41), (42). Namely, if a $V[0]$-valued function Ψ satisfies (41), (42), then for any j, the $V[0]$-valued function $(T_j)^2\Psi$ satisfies (41), (42).

7.3. **Monodromy of solutions with respect to shifts of z_j by 1.** Let $u^\xi(z_1,\dots,z_n,\lambda,\mu,\tau,p)$ be the function constructed for the tensor product $V = V_{\Lambda_1}\otimes\dots\otimes V_{\Lambda_n}$ in (30). Let $D(\mu,z,p)$ be the End $(V[0])$-valued function defined in (33). Set

$$\Psi^\xi(z,\lambda,\mu,\tau,p) \,=\, (1\otimes D(\mu,z,p))\, u^\xi(z_1,\dots,z_n,\lambda,\mu,\tau,p)\,.$$

According to Theorem 31, the $V[0]\otimes V[0]$-valued function Ψ^ξ is a solution of the qKZB equations with respect to the first factor. According to Corollary 18, for any j, the $V[0]\otimes V[0]$-valued function $\Psi^\xi_j(z,\lambda,\mu,\tau,p) = \Psi(\dots,z_j+1,\dots,\lambda,\mu,\tau,p)$ is a new solution of the same equations.

The next Proposition describes a relation between the two solutions and can be considered as a description of the monodromy of the hypergeometric solutions constructed in Section 6.4 with respect to shifts of variables z_j by 1.

For any j, introduce an entire function, ξ_j, of one variable by $\xi_j(\lambda) = \xi(\lambda-2\eta\Lambda_j)$.

Proposition 39. *For any j, we have*

$$\Psi^\xi_j(z,\lambda,\mu,\tau,p) \,=\, (1\otimes(D_j(\mu))^{1/p})\,\Psi^{\xi_j}(z,\lambda,\mu,\tau,p)$$

The Proposition follows from formula (31).

Remark. Assume that $\Lambda_1,\dots,\Lambda_n$ are natural numbers, then the square of the transformation

$$T_j : \Psi(z_1,\dots,z_n,\lambda) \,\to\, \Psi(z_1,\dots,z_j+1,\dots,z_n,\lambda)$$

preserves the theta function properties (41), (42). Namely, if a $V[0]$-valued function $\Psi(z,\lambda)$ satisfies (41), (42), then for any j, the $V[0]$-valued function $\Psi(\dots,z_j+2,\dots,\lambda)$ satisfies (41), (42).

Recall also that if $-p/4\eta = N$, f is a scalar theta function of level N and $\xi(\lambda) = f(\lambda/p)$, then $\Psi^\xi(z,\lambda,\mu,\tau,p)$ is a vector-valued theta function of level $m+N$. Notice now that if $\Lambda_1,\dots,\Lambda_n$ are natural numbers, then $\xi(\lambda-4\eta\Lambda_j) = f(\lambda/p+\Lambda_j/N)$, and $f(\lambda+\Lambda_j/N)$ is a new scalar theta function of level N.

References

[D] V. Drinfeld, *Quasi-Hopf Algebras*, Leningrad Math. J. 1 (1990), 1419–1457.

[F] G. Felder, *Conformal field theory and integrable systems associated to elliptic curves*, Proceedings of the International Congress of Mathematicians, Zürich 1994, p. 1247–1255, Birkhäuser, 1994; *Elliptic quantum groups,* preprint hep-th/9412207, to appear in the Proceedings of the ICMP, Paris 1994.

[FR] I. Frenkel and N. Reshetikhin, *Quantum affine algebras and holonomic difference equations*, Commun. Math. Phys. 146 (1992), 1–60.

[FTV] G. Felder, V. Tarasov and A. Varchenko, *Solutions of the elliptic qKZB equations and Bethe ansatz I,* preprint 1996, q-alg/9606005, to appear in the volume dedicated to V.I.Arnold's 60-th birthday.

[FV1] G. Felder and A. Varchenko, *On representations of the elliptic quantum group $E_{\tau,\eta}(sl_2)$*, Commun. Math. Phys. 181 (1996), 746–762.

[FV2] G. Felder and A. Varchenko, *Algebraic Bethe ansatz for the elliptic quantum group $E_{\tau,\eta}(sl_2)$*, Nuclear Physics B 480 (1996) 485-503.

[JM] M. Jimbo and T. Miwa, *Algebraic Analysis of Solvable Lattice Models*, Regional Conference Series in Mathematics, v. 85, AMS, 1995.

[K] T. Kohno, *Monodromy Representations of Braid Groups and Yang-Baxter Equations*, Ann. Inst. Fourier 37 (1987), 139–160.

[R] R. Randell, *Lattice Isotopic Arrangements are Topologically Isomorphic*, Proc. A.M.S. 107 (1989), 555–559.

[S] F. Smirnov, *Form Factors in Completely Integrable Models of Quantum Field Theory*, World Scientific, Singapore, 1992.

[TV1] V. Tarasov and A. Varchenko, *Geometry of q-hypergeometric functions as a bridge between Yangians and quantum affine algebras*, preprint q-alg 9604011, to appear in Invent. Math.

[TV2] V. Tarasov and A. Varchenko, *Geometry of q-Hypergeometric Functions, Quantum Affine Algebras and Elliptic Quantum Groups*, preprint q-alg/9703044.

[V] A. Varchenko, *Quantized Knizhnik–Zamolodchikov equations, quantum Yang–Baxter equations and difference equations for q-hypergeometric functions*, Commun. Math. Phys. 162 (1994), 499–528.